Based on National
Curriculum of Pakistan 2022-23

Model Textbook of
MATHEMATICS
Grade-9

National Book Foundation
as
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Islamabad

Based on National Curriculum of Pakistan 2022-23

Model Textbook of
Mathematics
Science Group
Grade
09

National Curriculum Council
Ministry of Federal Education and Professional Training

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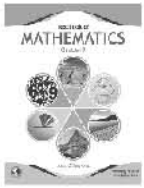
Model Textbook of **Mathematics**
for Grade 9



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Preface

This Model Textbook for Mathematics grade 9 has been developed by NBF according to the National Curriculum of Pakistan 2022-2023. The aim of this textbook is to enhance learning abilities through inculcation of logical thinking in learners, and to develop higher order thinking processes by systematically building upon the foundation of learning from the previous grades. A key emphasis of the present textbook is on creating real life linkages of the concepts and methods introduced. This approach was devised with the intent of enabling students to solve daily life problems as they go up the learning curve and for them to fully grasp the conceptual basis that will be built upon in subsequent grades.

After amalgamation of the efforts of experts and experienced authors, this book was reviewed and finalized after extensive reviews by professional educationists. Efforts were made to make the contents student friendly and to develop the concepts in interesting ways.

The National Book Foundation is always striving for improvement in the quality of its books. The present book features an improved design, better illustration and interesting activities relating to real life to make it attractive for young learners. However, there is always room for improvement and the suggestions and feedback of students, teachers and the community are most welcome for further enriching the subsequent editions of this book.

May Allah guide and help us (Ameen).

Dr. Raja Mazhar Hameed
Managing Director

بِسۡمِ اللّٰہِ الرَّحۡمٰنِ الرَّحِیۡمِ

اللہ کے نام سے شروع جو بڑا مہربان، نہایت رحم والا ہے

Content

UNIT
01

REAL NUMBERS

In this unit the students will be able to:

- Recall the history of numbers.
- Recall the set of real numbers as a union of sets of rational and irrational numbers.
- Depict real numbers on the number line.
- Demonstrate a number with terminating and non-terminating recurring decimals on the number line.
- Give decimal representation of rational and irrational numbers.
- Know the properties of real numbers.
- Explain the concept of radicals and radicands.
- Diferentiate between radical form and exponential form of an expression.
- Transform an expression given in radical form to an exponential form and vice versa.
- Recall base, exponent and value.
- Apply the laws of exponents to simplify expressions with real exponents.

A number is an abstract idea used in *counting* and *measuring*. A symbol which represents a number is called a numeral, but in common usage the word number is used for both the idea and the symbol. In addition to their use in counting and measuring, numerals are often used for labels (telephone numbers), for ordering (serial numbers), and for codes (ISBN, i.e. International Standard Book Number). In mathematics, the definition of number has been extended over the years to include such numbers as zero, negative numbers, rational numbers, irrational numbers, real numbers and complex numbers.

1.1 History of Real Numbers

Can you imagine a world without numbers?

In our daily conversations, our domestic activities and at our jobs, we cannot spend a single day without using numbers. Our lives will be quite strange without involvement of numbers. In this way one can imagine about the life of a person in the era when numbers were not discovered.

There is an interesting story regarding how humans started using numbers for the first time. The story is about a shepherd boy who used pebbles to count his goats /sheep he sent for grazing each day to avoid missing any. The number discovered in that way were simply the counting numbers which are now called **Natural numbers**. People usually used their fingers or sticks to count objects and were symbolized by tally marks. They counted objects by carving tally marks into cave walls, bones, woods or stones about 30000 BC. In fact, the earliest numerals recorded so far were simple marks for small numbers and special symbol for 10. These symbols appeared in early Egyptians inscriptions around 3500 BC to 3000 BC.

Sumerians Contribution to Numeral System

Some historians believe that Sumerians were first to use symbols for numerals

Around 5000 BC. Sumerian was a Great civilization settled at the Fertile Crescent area near present Iraq. They had a great contribution to the development of number system and basic Mathematics as they are considered to be very advanced in many fields at that era. e.g.

> **Interesting Information**
>
> The first record of existence of tally marks is on a leg bone of a baboon dating prior to 30000 BC. The bone has 29 clear notches in a row. It was discovered in South Africa.

- They were first to construct buildings.
- They initiated modern agriculture, so they were keen about fundamental calculations.
- They depended on rise and fall of sun to estimate time, which showed that they were keen observers of angles and geometry.
- They used cuneiforms on clay tablets.

Babylonians Contribution to Numeral System

Babylonians system of numerals based on Earlier Sumerian numeral system basically, as they were descendants of Sumerian. They relied upon a series of cuneiform marks to represent digits. This was a sexagesimal system of numbers (system with base 60). This concept is still in use today as in division of time we use 60 minutes, 60 seconds etc.

Although they carried complicated algebraic calculations and knew about the concept of nothingness but they didn't symbolize it ever rather they used a space between digits to represent zero. It made their numbers and calculations quite ambiguous. Around Babylon, clay was abundant so they made a lot of use of clay tablets impression with cuneiforms. The cuneiforms and numerals occur together in some documents from about 3000 BC. They seem to have some convention regarding the use. Cuneiform was always used for wages due while wages paid were written in curvilinear.

Greek Contribution to Numeral System

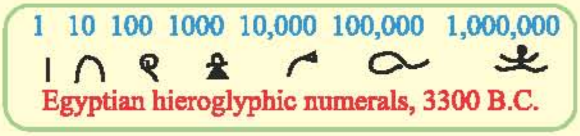
The early Greeks, like their predecessors Egyptians and Babylonians, also repeated units to 9 and probably had " – " symbols for ten and " O " for 100. The Greek system of abbreviation in numbers called Attick numerals is present in the record of 5th century BC but probably was used much earlier. Like Babylonians and Romans, the ancient Greeks knew about nothingness but did not symbolize the concept.

Researchers had discovered that many other civilizations developed positional notation of numbers independently including the Ancient Chinese and Aztecs.

Romans Contribution to Numeral System

Romans used tally marks on tally sticks or tally bones. Like early humans they also used V to represent five and X for ten. Ancient Romans incorporated these symbols into their seven symbol system. The Roman empire was very vast and this system of numerals was used thus widely throughout Europe from early 2000 years ago to late middle ages. Like the Babylonians, the Ancient Roman Numeral system lacked to symbolize nothingness. This system was maintained for nearly 2000 years in commerce and scientific literature.

i.



Egyptian hieroglyphic numerals, 3300 B.C.

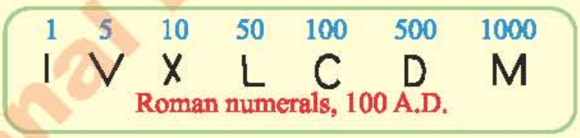
ii.

Babylonian cuneiform numerals, 2000 B.C.

iii.

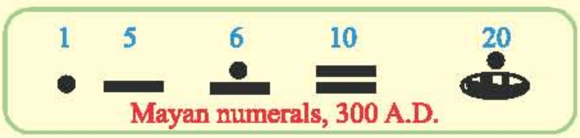
Early Greek numerals, 5 B.C.

iv.



Roman numerals, 100 A.D.

v.



Mayan numerals, 300 A.D.

vi.

0 1 2 3 4 5 6 7 8 9

Hindu-Arabic digits, present day

Discovery of Zero (the sooper Hero)

Historians believed that Mayans living in central America used the idea of zero in their calendar system but they were isolated from other world so it couldnot caoe in outer world. Some historians give the crown of symbolizing nothingness to the Indian mathematician and astronomer Brahmagupta in 628 AD. It is also narrated that a great Muslim Mathematician Abu Muhammad Musa al Khwarizmi, who was also an astronomer and a geographer, contributed much to our modern understanding of maths. He described a number system based on 10 numerals from zero to nine in 7th century in his book 'The use of the Hindu Numerals'(كتاب فی استعمال الاعداد الهندی) . He called this new digit as 'Sifr'. This useful system was soon adopted by Arabs.

Zero in the Europe

Fibonacci, the son of an Italian merchant discovered that Arab traders were using accounting system based on 0-9 numerals. He quickly understood its importance and improved book keeping and accounting in Europe. In 1202, he published a book describing this number system. He elaborated in the book about practical application i.e. how to convert one currency into other, calculation of profit and losses and other important business needs. In Italy 'Sifr' became 'zefero' which later became zero.

Discovery of zero brought a new set of numbers called set of whole numbers and it reduced the hurdles in calculation and understanding numbers.

Negative Numbers

The Chinese Mathematician Diophantus was most probably the first who used negative numbers around 200 BC in his work. He represented the amount of debt or loss. Then in 7th century the Indian Brahmagupta wrote rules for adding, subtracting, multiplying and dividing negative numbers. The discovery of negative numbers gave existence to the set of Integers.

Rational Numbers

Pythagoras the Greek mathematician used fractions for the first time in mathematics around 500BC, which was infact the discovery of rational numbers. The word rational came from ratio.

Irrational Numbers

Soon after the discovery of rational numbers by Pythagoras, one of his early follower, Hippasus of Metapontum was working to find the hypotenuse of a right isosceles triangle with 2 equal sides of length 1 unit. He came with a strange answer ($\sqrt{2}$) and concluded that the answer is not reasonable number because it cannot be written in a/b form. He called such numbers as irrational (meaning stupid, nonsense or not reasonable). It is narrated that Hippasus was drowned in the sea for his new concept of numbers as it was against their self made religion.

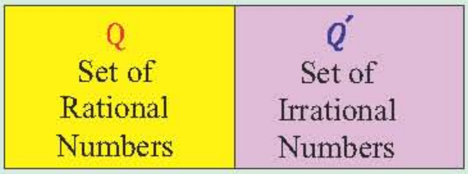
Real numbers

The set of real numbers was thus completed with the discovery of rational and irrational numbers. We know today that the set of reals contain both and there is no such real number which is neither rational nor irrational.

1.2 Introduction to Real Numbers

Earlier mathematicians particularly Richard Dedekind have precisely defined the concept of real numbers which include both rational numbers such as $\frac{2}{9}$ and irrational numbers such as $\sqrt{2}$.

The real numbers are used to solve real life problems such as finding velocity, speed or distance, the profit or loss of a business, the difference in stock market etc.

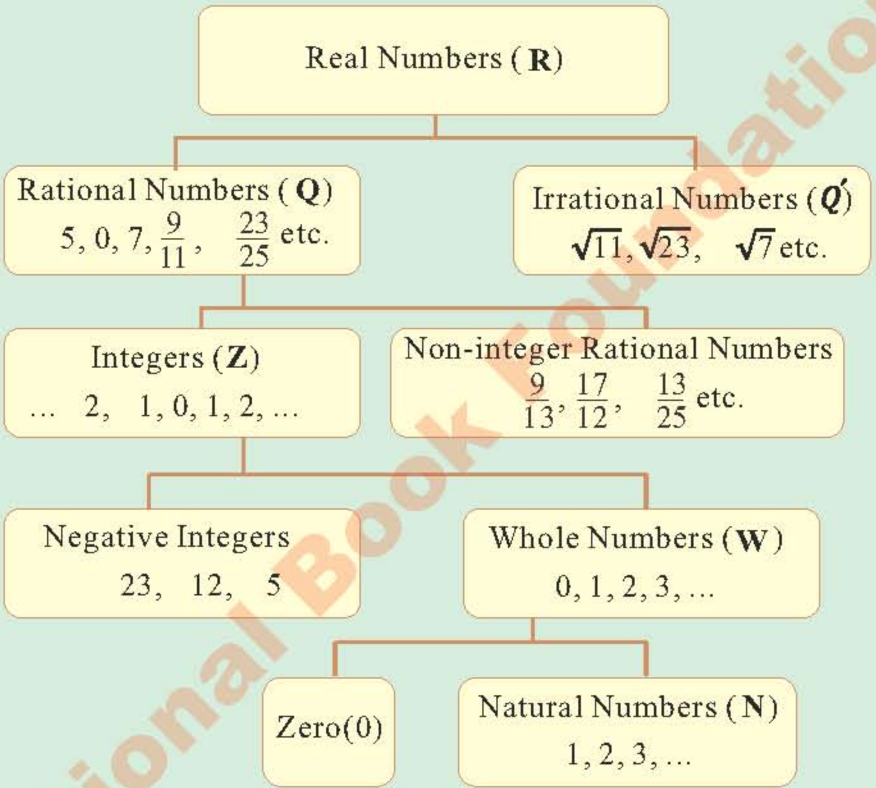


$R = Q \cup Q'$

$Q \cap Q' = \phi.$

Key Fact

Set of rational numbers (Q) and set of irrational numbers (Q') are disjoint i.e., $Q \cap Q' = \phi$. But $R = Q \cup Q'$, then Q and Q' are called exhaustive sets.



1.2.1 Number Line

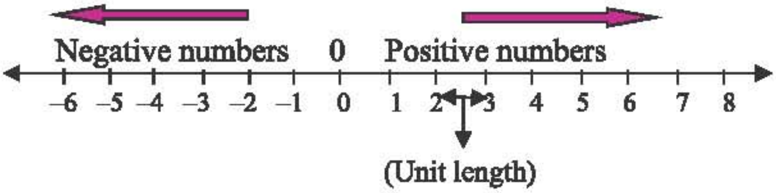
We use a number line to visualize real numbers and their relation to each other. To construct a number line, we choose a point corresponding to the number '0'. Points at equally spaced intervals are then associated with the integers. The positive integers are to the right of '0' and negative integers are to the left of 0. All other real numbers are associated with a point which is called the *coordinate* **of that point**. The point associated with zero is called the *origin*.

Enlighten Yourself

Number line is also called a real line because we express real numbers on it.

Usually few integers are shown on a number line to indicate the unit length of the line, that is, the distance between any two consecutive integers.

A number line is shown in figure below.



1.3 Properties of Real Numbers

The basic properties of real numbers are w.r.t addition and multiplication. In this section, some of the properties of these operations are reviewed. The following results are true for any real numbers *a*, *b* and *c*.

Name of the property	With respect to		Examples	
	+	×	+	×
Closure	$a+b\in R$	$a\bullet b\in R$	$6+4=10\in R$	$6\times4=24\in R$
Commutative	$a+b=b+a$	$a\bullet b=b\bullet a$	$4+7=7+4$ $=11$	$4\times7=7\times4$ $=28$
Associative	$a+(b+c)=(a+b)+c$	$a\bullet(b\bullet c)=(a\bullet b)\bullet c$	$4+(6+8)=(4+6)+8$ $=18$	$4\times(6\times8)=(4\times6)\times8$ $=192$
Identity	$a+0=a=0+a$	$a\bullet1=1\bullet a=a$	$6+0=0+6$ $=6$	$6\times1=1\times6$ $=6$
Inverse	$a+(-a)=-a+a$ $=0$	$a\bullet\frac{1}{a}=\frac{1}{a}\bullet a=1$ $a\neq0$	$14+(-14)=-14+14$ $=0$	$14\times\frac{1}{14}=\frac{1}{14}\times14$ $=1$

Key Fact

i. 0 is the additive identity and 1 is the multiplicative identity of real numbers.

ii. $-a$ is the additive inverse of a and $\frac{1}{a}$ $(a\neq0)$ is the multiplicative inverse of a.

1.3.1 Distributive Property of Multiplication over Addition

The distributive property involves both operations i.e., addition and multiplication.
Distributive property says, for all real numbers a, b, and c:

$$a(b+c)=ab+ac$$

Thinking Corner

- Which number has the additive inverse the number itself.
- Do all real numbers have their multiplicative inverses.
- Which number has no multiplicative inverses.

Example 1: If $a=5$, $b=6$ and $c=9$, then verify that
$a(b+c)=ab+ac$.

Solution:
$a(b+c)=5(6+9)=5(15)=75$
$ab+ac=5(6)+5(9)=30+45=75$

Thus:

$$a(b+c)=ab+ac$$

Example 2: Use the distributive property to simplify

$$32\left(\frac{1}{8}+\frac{1}{4}\right).$$

Solution: $32\left(\frac{1}{8}+\frac{1}{4}\right)=32\times\frac{1}{8}+32\times\frac{1}{4}$ $\quad a\,(b+c)=ab+ac$

$=4+8=12$

Enlighten Yourself

Distributive property of multiplication over subtraction, $a\,(b-c)=ab-ac$ also holds.

1.3.1 Properties of Equality and Inequality of Real Numbers

Equality and Inequality Symbols

There are three symbols which can be used to show the possible relations between any two real numbers 'a' and 'b'. These are <, = and >, where '=' is equality symbol and '< and >' are inequality symbols.

Following table shows the use of these symbols.

Read	Write
a is less than *b*	$a<b$
a is equal to *b*	$a=b$
a is greater than *b*	$a>b$

Historya Mystery

The symbol < is used for "is less than" and > is used for "is greater than". These were introduced by Thomas & Harriot around 1630.

If a and b are real numbers, then only one of the following statement is true

(i) $a<b$ (ii) $a=b$ (iii) $a>b$

This property is known as **Trichotomy Property.**

A mathematical statement with the equality sign is called an '**equality**'. A mathematical statement in which we do not use the symbol of equality but other symbols like '<' or '>' or both, is called an '**inequality**'.

Properties of Equality of Real Numbers

The following properties are true for any real numbers a, b and c.

Name of property	General statement
Reflexive	$a=a$
Symmetric	If $a=b$ then $b=a$
Transitive	If $a=b$ and $b=c$ then $a=c$
Additive	If $a=b$ then $a+c=b+c$

Multiplicative	If $a = b$ then $ac = bc$ or $ca = cb$, where $c \neq 0$
Cancellation w.r.t. addition	If $a + c = b + c$ then $a = b$
Cancellation w.r.t. multiplication	If $ac = bc$ then $a = b$ If $ca = cb$ then $a = b$ } where $c \neq 0$

Properties of Inequality of Real Numbers

The following properties are true for any real numbers *a*, *b* and *c*.

Name of property	General statement	Examples:
Trichotomy	Either $a > b$ or $a = b$ or $a < b$	If $2 < 3$, then $2 \neq 3$ and $2 \not> 3$
Transitive	If $a < b$ and $b < c$ then $a < c$ If $a > b$ and $b > c$ then $a > c$	If $3 < 5$ and $5 < 7$, then $3 < 7$ If $-2 > -5$ and $-5 > -7$, then $-2 > -7$
Additive	If $a < b$, then $a + c < b + c$ If $a > b$, then $a + c > b + c$	If $3 < 5$, then $3 + 10 < 5 + 10$ If $-5 > -8$, then $-5 + 2 > -8 + 2$
Multiplicative	(a) For $c < 0$ and $a, b \in R$, (i) If $a > b$ then $ac < bc$ (ii) If $a < b$, then $ac > bc$	(a) For $-4 < 0$, (i) If $3 > 2$, then $3(-4) < 2(-4)$ $-12 < -8$ (ii) If $3 < 5$, then $3(-4) > 5(-4)$ $-12 > -20$
Multiplicative	(b) For $c > 0$ and $a, b \in R$, (i) If $a > b$, then $ac > bc$ (ii) If $a < b$, then $ac < bc$ (iii) If $a < b$, then $\frac{1}{a} > \frac{1}{b}$	(b) For $3 > 0$ (i) If $5 > 2$, then $5(3) > 2(3)$ $15 > 6$ (ii) If $2 < 3$, then $2(5) < 3(5)$ $10 < 15$ (iii) If $2 < 3$, then $\frac{1}{2} > \frac{1}{3}$
Cancellation w.r.t addition	If $a + c < b + c$ then $a < b$ If $a + c > b + c$ then $a > b$	If $2 + 3 < 5 + 3$ then $2 < 5$ If $4 + 3 > 2 + 3$ then $4 > 2$
Cancellation w.r.t multiplication	If $ac < bc$ and $c > 0$ then $a < b$ if $ac < bc$ and $c < 0$ then $a > b$ if $ac > bc$ and $c > 0$ then $a > b$ if $ac > bc$ and $c < 0$ then $a < b$	If $2 \times 3 < 4 \times 3$ then $2 < 4$ If $-3 \times 2 > -3 \times 4$ then $2 < 4$

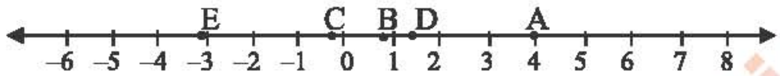
Key Fact

- The inequality sign remains unchanged if a positive number is multiplied to both the sides of an inequality.
- The inequalities are reversed if a negative number is multiplied to both the sides of an inequality.

Example 3: Show the following numbers on a number line.

(a) 4 (b) $\frac{7}{8}$ (c) $-\frac{1}{3}$ (d) $\sqrt{2}$ (e) $-\pi$

Solution:



(a) The number 4 is four units to the right of **0**, therefore, A is representing 4 on number line.

(b) $\frac{7}{8}$ = 0.875 is between 0 and 1, which is a terminating decimal. Point B in the figure is representing $\frac{7}{8}$ on the number line.

(c) $-\frac{1}{3} = -0.333\ldots$ or $-0.\bar{3}$ is between 0 and –1, which is a recurring decimal. Point C in the figure is representing $-\frac{1}{3}$ on the number line.

(d) Since $\sqrt{2} = 1.414213\ \ldots$, is between 1 and 2. Point D in the figure is the location of $\sqrt{2}$.

(e) Since $-\pi = -3.14159\ldots$ is between –3 and –4. Point E in the figure is the location of $-\pi$.

1.3.2 Representation of Real Numbers on Number Line

The representation of real numbers on a number line is called graphing the real numbers or graph of the real numbers.

Example 4: Represent the following sets of real numbers on a number line.

(a) $x \leq 2$ (b) $-6 < x < 4$

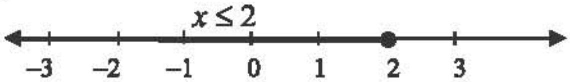
(c) $x > -4$ (d) $-2 \leq x < 1$

Thinking zone

Try to imagine the numbers less than or equal to 2 and relate the words at least or at most which ever suitable in this case.

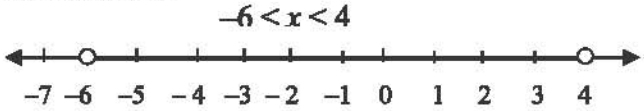
Solution:

(a) The inequality $x \leq 2$ specifies all real numbers less than or equal to 2. This set is represented on a real number line as follows.



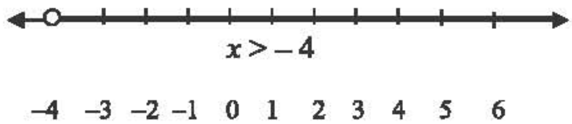
A filled circle indicate that 2 is included in the set.

(b) The inequality $-6 < x < 4$ specifies all real numbers between – 6 and 4, as shown on the number line.



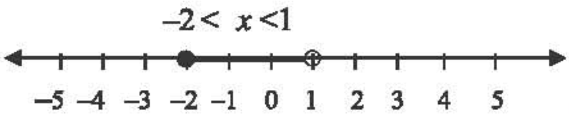
We use hollow circle to indicate that both –6 and 4 are not included in the set.

(c) $x > -4$ specifies all real numbers greater than -4.



We use hollow dot to indicate that – 4 is not included in the set.

(d) $-2 \le x < 1$ is the set of all real numbers between –2 and 1 including –2 but excluding 1.



EXERCISE 1.1

1. Represent each number on the number line.

(i) $\frac{3}{4}$ (ii) $\frac{-1}{3}$ (iii) $4\frac{1}{2}$ (iv) $-\sqrt{8}$

(v) $\sqrt{8}$ (vi) $-4\frac{1}{2}$ (vii) $\frac{1}{3}$ (viii) $-\frac{7}{8}$

2. Identify the property that justifies.

(i) $1 \times (y-2) = y-2$ (ii) $(0.2)\,5 = 1$

(iii) $(x+2)+y = y+(x+2)$ (iv) $-(3b)+(3b) = 0$

(v) $(x+5)-1 = x+(5-1)$ (vi) $-3\,(2-y) = -6+3y$

3. Represent the following on a number line.

(i) $x < 0$ (ii) $-3 < x < 3$ (iii) $x \ge -8$

(iv) $x > 0$ (v) $x < -3$ (vi) $-4 < x \le 4$

4. Identify the properties of equality and inequality of real numbers that justifies the statement.

(i) $9x = 9x$ (ii) If $x+2 = y$ and $y = 2x-3$, then $x+2 = 2x-3$

(iii) If $2x+3 = y$, then $y = 2x+3$ (iv) If $3 < 4$, then $-3 > -4$

(v) If $2y + 2w = p$ and $p = 50$, then $2y + 2w = 50$

(vi) If $x+4 > y+4$, then $x > y$ (vii) If $2 < 5$ and $5 < 9$, then $2 < 9$

(viii) If $-18 < -16$, then $9 > 8$

1.4 Radical and Radicands

1.4.1 Square Root

A square root of a positive number 'n' is another number 'm' whose square is 'n.' Any positive number has two square roots, which are additive inverses of each other.

For example, 4 is a square root of 16, because $(4)^2 = 16$ and -4 is also a square root of 16, because $(-4)^2 = 16$. Therefore, the two square roots of 16 are -4 and 4, which are additive inverse of each other.

1.4.2 Principal Square Root

Positive square roots are called 'principal square roots'. e.g. $\sqrt{25} = 5, \sqrt{81} = 9$ etc.

In expressions like $\sqrt{25}$ entire $\sqrt{25}$ is called a **square radical or radical**. The symbol $\sqrt{\ }$ is called a **radical sign** and the number '25' under the radical sign is called the **radicand**.

Definition of n^{th} Root

For any real numbers 'a' and 'b' and any positive integer $n > 1$, if $a^n = b$, then $a = b^{\frac{1}{n}}$, where 'n' is the index of the radical.

We read $a^n = b$ as 'b is the n^{th} power of a' and $a = b^{\frac{1}{n}}$ as 'a is the n^{th} root of b'.

For example, $y^3 = x$, then $y = x^{\frac{1}{3}} = \sqrt[3]{x}$

Here 3 is the index of radical and y is the cubic root of x.

Example 5:

Radical Form	Index of the Radical	Radicand
$\sqrt[3]{35}$	3	35
$\sqrt[5]{\frac{xy}{z}}$	5	$\frac{xy}{z}$
$\sqrt{-(xyz)^4}$	2	$-(xyz)^4$

Historya Mystery

The radical sign was first used in 1525 AD and was written as "$\surd$".

Key Fact

(i) $\sqrt{b} = \sqrt[2]{b}$ i.e. $\sqrt{\ }$ and $\sqrt[2]{\ }$ are equivalent.

(ii) There is no real number that is a square root of a negative number.

e.g. $\sqrt{-16} \neq 4$, since $(+4)^2 \neq -16$. In Mathematics 'imaginary numbers' are defined to handle the square root of negative numbers.

1.4.3 Properties of Radicals

1. Product and Quotient Rules for Radicals

For any integer n > 1 and for all real numbers 'a' and 'b' for which the operations are defined

(i) $\sqrt[n]{a}.\sqrt[n]{b} = \sqrt[n]{ab}$ ⟶ Product rule for radicals

e.g. $\sqrt[3]{8} \times \sqrt[3]{27} = \sqrt[3]{8\times 27}$

(ii) and $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$ ⟶ Quotient rule for radicals e.g. $\dfrac{\sqrt[3]{64}}{\sqrt[3]{8}} = \sqrt[3]{\dfrac{64}{8}}$

Example 6: Use the product rule for radicals to simplify the following. Assume that all variables represent positive numbers.

(a) $\sqrt{2a}.\sqrt{7b}$ (b) $\sqrt[4]{\dfrac{1}{x}}.\sqrt[4]{\dfrac{2}{y}}$ (c) $\sqrt[3]{3}.\sqrt{2}$

Solution:

(a) $\sqrt{2a}.\sqrt{7b} = \sqrt{14ab}$

(b) $\sqrt[4]{\dfrac{1}{x}}.\sqrt[4]{\dfrac{2}{y}} = \sqrt[4]{\dfrac{2}{xy}}$

(c) The product rule for radicals does not apply to $\sqrt[3]{3}.\sqrt{2}$, because the indices are not same.

Key Fact

- The product and quotient rules for radicals apply only if the indices are same.
- The plural of index is indices.

2. Reducing the Index

If the index of the radical and the exponent of the radicand have a common factor, the expression can be written with a smaller index.

We will explain it with the help of an example. Consider $\sqrt[12]{9^6}$, here the index of the radical is 12 and exponent of radicand is 6. Also 6 is the common factor of 12 and 6. So we can write the expression with a smaller index as follows.

$$\sqrt[12]{9^6} = 9^{\frac{6}{12}} = 9^{\frac{1}{2}} = \sqrt{9} = 3$$

Example 7: Write the given expression with a smaller index. Assume that variable t represents positive numbers.

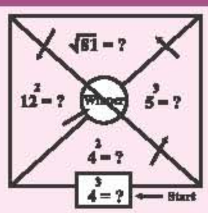
$$\sqrt[15]{t^{10}}$$

Solution : $\sqrt[15]{t^{10}} = t^{\frac{10}{15}}$

$= t^{\frac{2}{3}} = \sqrt[3]{t^2}$

Math Play Ground

1. Make a hopscotch on ground as give:
2. Ask a player to start hoping and doing sums in each box.
3. If reaches in circle without falling and doing correct calculations, he/she wins.



1.5 Laws of Exponents / Indices

1.5.1 Base and Exponents

We use exponent to indicate the repeated multiplication of the same factors.
The exponent indicates that how many times a factor, called the *base,* occurs in the multiplication form.

e.g.

$$3.3.3.3 = 3^4$$

Exponent (4) — Base (3)

The expression 3^4 is an exponential form read as 'three to the fourth power', where as 3.3.3.3 is the factored form. The words 'square' and 'cube' are sometimes used for exponents '2' and '3' respectively, rather than 'to the second power' and 'to the third power'.

Historya Mystery

Rene Descartes (1596 – 1650) was the first mathematician who extensively used exponential notation as it is used today. However, for some unknown reason, he always used xx for x^2.

Key Fact

When the exponent is a natural number, the base can be any real number.
We use an exponent as a convenient way to write repeated multiplication.

1.5.2 Rational Exponents

When we perform operations with exponents, we have to define a zero exponent and a negative exponent. This may lead us to define a rational exponent.

Definition of $b^{\frac{1}{n}}$

If b is a real number and n is a positive integer, then $b^{\frac{1}{n}} = \sqrt[n]{b}$ e.g. $8^{\frac{1}{3}} = \sqrt[3]{8}$.

Definition of $b^{\frac{m}{n}}$

If m and n are positive integers with no common factor except 1 and $n \neq 0$, then
$b^{\frac{m}{n}} = \left(b^{\frac{1}{n}}\right)^m = (\sqrt[n]{b})^m$ for all real numbers b. e.g. $36^{\frac{3}{2}} = \left(36^{\frac{1}{2}}\right)^3$.

We can use this definition to write expressions with rational exponents as radicals.
The number 'n' indicates the index of the radical and the number 'm' indicates the power to which the radical is to be raised.

The procedure for evaluating $b^{\frac{m}{n}}$ can be summarized as follows.

I. Determine the nth root of b.
II. Raise the result to the m power.

Example 8: In the following table:

(a) perform the operation in column-A and compare the result to the value of radical in column- B.

(b) what do you observe about the denominator of the exponent and the index of the radical?

	Column-A (exponential form)	Column-B (radical form)
(i)	$9^{\frac{1}{2}}$	$\sqrt{9}$
(ii)	$64^{\frac{1}{3}}$	$\sqrt[3]{64}$
(iii)	$81^{\frac{1}{4}}$	$\sqrt[4]{81}$
(iv)	$32^{\frac{1}{5}}$	$\sqrt[5]{32}$

Solution:

(a)

(i) $9^{\frac{1}{2}} = 3$ and $\sqrt{9} = 3$

(ii) $64^{\frac{1}{3}} = (4^3)^{\frac{1}{3}} = 4$ and $\sqrt[3]{64} = 4$

(iii) $81^{\frac{1}{4}} = 3$ and $\sqrt[4]{81} = (3^4)^{\frac{1}{4}} = 3$

(iv) $32^{\frac{1}{5}} = 2$ and $\sqrt[5]{32} = 2$

(b) In each part, the exponential and the radical expression have the same value. The denominator of the exponent is the same as the index of the radical.

Example 9: Simplify:

(a) $8^{\frac{2}{3}}$ (b) $36^{\frac{3}{2}}$

Solution:

(a) $8^{\frac{2}{3}} = \left(8^{\frac{1}{3}}\right)^2 = \left[\left(2^3\right)^{\frac{1}{3}}\right]^2 = (2)^2 = 4$

(b) $36^{\frac{3}{2}} = \left(36^{\frac{1}{2}}\right)^3 = \left[\left(6^2\right)^{\frac{1}{2}}\right]^3 = (6)^3 = 216$

Math Play Ground

Jump on the numbers which are squares of natural numbers to go out.

IN

81	35	47	63	40
49	36	25	28	32
74	15	4	100	9
20	13	6	89	1

OUT

Negative Rational Exponents

For integral exponents, we define:

$a^{-n} = \frac{1}{a^n}$ provided $a \neq 0$. e.g. $8^{-3} = \frac{1}{8^3}$.

We can extend this definition to negative rational exponents.

If m and n are any two integers such that one of them is negative and they have no common factor other than 1 and if $b \neq 0$, then $b^{\frac{-m}{n}} = \frac{1}{b^{\frac{m}{n}}}$ for all $b \in R$, for which $b^{\frac{m}{n}}$ is defined.

Example 10: Simplify.

(a) $16^{\frac{-3}{4}}$ (b) $\left(\frac{16}{25}\right)^{\frac{-1}{2}}$

Solution:

(a)
$$16^{\frac{-3}{4}} = \frac{1}{(16)^{\frac{3}{4}}} = \frac{1}{(2^4)^{\frac{3}{4}}} = \frac{1}{2^3} = \frac{1}{8}$$

(b)
$$\left(\frac{16}{25}\right)^{\frac{-1}{2}} = \left(\frac{25}{16}\right)^{\frac{1}{2}} = \sqrt{\frac{16}{25}} = \sqrt{\frac{5^2}{4^2}} = \frac{5}{4}$$

Example 11: Write exponential expressions as an equivalent radical expression.

(a) $(-7)^{\frac{2}{3}}$ (b) $(2)^{\frac{3}{5}}$

Solution: (a) $(-7)^{\frac{2}{3}} = (-7)^{2\times\frac{1}{3}}$
$= (49)^{\frac{1}{3}}$
$= \sqrt[3]{49}$

(b) $(2)^{\frac{3}{5}} = \left[(2)^3\right]^{\frac{1}{5}}$
$= \sqrt[5]{8}$

1.5.3 Properties of Exponents

If m and n are rational numbers, then for non zero real numbers a and b for which the expressions are defined, the following are the properties of exponents.

i) $a^m . a^n = a^{m+n}$ → Product rule

ii) $\frac{a^m}{a^n} = a^{m-n}$ → Quotient rule

iii) $a^{-n} = \frac{1}{a^n}$ → Definition of negative exponent,

iv) $(a^m)^n = a^{mn}$ → Power of a power rule

v) $(ab)^n = a^n b^n$ → Power of a product rule

vi) $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ → Power of a quotient rule

vii) $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$ → Negative power of a quotient rule

Example 12: Use the properties of exponents to evaluate each of the following.

(a) $(5^6)^{\frac{1}{2}}$ (b) $2^{\frac{1}{2}} \cdot 8^{\frac{1}{2}}$

Solution:

(a) $\left(5^6\right)^{\frac{1}{2}}$

$= 5^{6\times\frac{1}{2}} = 5^3$

$= 125$

(b) $2^{1/2} \cdot 8^{1/2}$

$= (2 \cdot 8)^{\frac{1}{2}} = (16)^{\frac{1}{2}}$

$= (4^2)^{\frac{1}{2}} = 4$

Example 13: Use properties of exponents to simplify each of the following. Assume that all variables represent positive numbers. (Write all results with positive exponents.)

(a) $a^{\frac{1}{3}}(a^{\frac{5}{3}} - a^{\frac{-2}{3}})$

(b) $\left[\dfrac{x^{\frac{1}{2}}}{y^3}\right]^{-\frac{1}{3}}$

Solution: (a) $a^{\frac{1}{3}}(a^{\frac{5}{3}} - a^{\frac{-2}{3}})$

$= a^{\frac{1}{3}}a^{\frac{5}{3}} - a^{\frac{1}{3}}a^{\frac{-2}{3}}$

$= a^{\frac{1}{3}+\frac{5}{3}} - a^{\frac{1}{3}-\frac{2}{3}}$

$= a^{\frac{6}{3}} - a^{\frac{-1}{3}}$

$= a^2 - \dfrac{1}{a^{\frac{1}{3}}} = \dfrac{a^{\frac{2}{3}} - 1}{a^{\frac{1}{3}}}$

(b) $\left[\dfrac{x^{\frac{1}{2}}}{y^3}\right]^{-\frac{1}{3}}$

$= \left[\dfrac{y^3}{x^{\frac{1}{2}}}\right]^{\frac{1}{3}} = \dfrac{\left(y^3\right)^{\frac{1}{3}}}{\left(x^{\frac{1}{2}}\right)^{\frac{1}{3}}}$

$= \dfrac{y}{x^{\frac{1}{6}}}$

EXERCISE 1.2

1. By using the property of product and quotient rule for radicals, write each expression as a single radical and simplify.

(i) $\sqrt[3]{6} \cdot \sqrt[3]{6}$ (ii) $\sqrt[5]{4} . \sqrt[5]{8}$ (iii) $\sqrt[4]{x} . \sqrt[4]{x^3}$

(iv) $\sqrt{10} \cdot \sqrt[3]{11}$ (v) $\dfrac{\sqrt[4]{x^7}}{\sqrt[4]{x^5}}$ (vi) $\dfrac{\sqrt[3]{5000}}{\sqrt[3]{5}}$

(vii) $\dfrac{\sqrt[2]{500}}{\sqrt[2]{5}}$ (viii) $\sqrt[2]{10} \cdot \sqrt[3]{7}$

2. Write each exponential expression as an equivalent radical expression and simplify if possible.

(i) $(216)^{\frac{2}{3}}$ (ii) $(29)^{\frac{1}{2}}$ (iii) $\left(\dfrac{1}{32}\right)^{\frac{1}{5}}$

(iv) $(216)^{-\frac{2}{3}}$ (v) $(1000)^{\frac{1}{3}}$ (vi) $\left(\dfrac{1}{39}\right)^{-\frac{1}{2}}$

3. Write each radical expression as an equivalent exponential expression and simplify if possible.

(i) $\left(\sqrt[3]{5}\right)^2$ (ii) $\left(\sqrt[4]{10}\right)^8$ (iii) $-\left(\sqrt[3]{6}\right)^6$

(iv) $\left(\sqrt[3]{6}\right)^6$ (v) $-\left(\sqrt[3]{5}\right)^2$ (vi) $-\left(\sqrt[4]{10}\right)^8$

4. Use the properties of exponents to simplify each of the following. Assume that all variables represent positive numbers. (write all results with positive exponents.)

(i) $\dfrac{16^{\frac{1}{5}}.16^{\frac{1}{4}}}{16^{\frac{-3}{10}}}$ (ii) $7^{\frac{-1}{3}}\left(7^{\frac{5}{3}} - 7^{\frac{4}{3}}\right)$ (iii) $\dfrac{2^{\frac{2}{3}}.2^{\frac{1}{7}}}{2^{\frac{1}{2}}}$

(iv) $\dfrac{3^{\frac{-1}{2}}.3^{\frac{1}{2}}}{3^{\frac{1}{2}}}$ (v) $\left(\dfrac{36^{\frac{1}{2}}.}{8^{\frac{1}{2}}.}\dfrac{6^{\frac{1}{2}}}{27^{\frac{1}{2}}}\right)^3$ (vi) $\left(\dfrac{2187\ a^5 b^{17}}{a^{12} b^3}\right)^{\frac{1}{7}}$

(vii) $\sqrt[4]{\dfrac{a^3}{b^3}} \times \sqrt[4]{\dfrac{b^3}{c^3}} \times \sqrt[4]{\dfrac{c^3}{a^3}}$

5. Use suitable laws of exponents to show that

$$\left(\frac{x^p}{x^q}\right)^{p+q} \times \left(\frac{y^q}{y^r}\right)^{q+r} \times \left(\frac{z^r}{z^p}\right)^{r+p} \times x^{q^2} \times y^{r^2} \times z^{p^2} = x^{p^2} \times y^{q^2} \times z^{r^2}$$

1.5.4 Application of Real Numbers in Daily Life

All the numbers we use in our daily life situations are Real numbers. We cannot imagine life without numbers. For instance we use natural numbers in counting our objects in the pantry, books in the library, animals or birds at a farm, stock taking in inventory of a factory etc. Similarly, we have a vast use of integers while recording or understanding temperature, gain or loss, rise or fall etc. Rational numbers have also the vast contribution in daily life situations such as use of ratio, proportion, fractions and percentages in financial matters like income, expenditure, savings, and payment of wages to employees, rents of buildings, profit, loss sharing in business managements, risk calculations. The irrational numbers as obvious from the name are not reasonable or they don't make a sense for non-mathematicians. But for mathematicians they have really big scope of usage. Engineers, technicians, opticians while working with circles, spheres or cylinders and finding their areas, perimeters or volume, which include π are working with irrational numbers. Then we find irrational numbers like in architecture, navigation and fluid mechanics, where transcendental functions are in frequent use.

Example 14:

A cooking oil company produces four types of oils in packing of 1 litre , 5 litre & 10 litre. The inventory is shown in the table.

Name / packing size	1 litre	5 litre	10 litre
Cooking oil-I	5000	2500	1000
Cooking oil-II	5000	2500	1000
Cooking oil-III	5000	2500	1000
Cooking oil-IV	5000	2500	1000

After removal of 40% of an item, it is to be replenished. The daily removal of 1lire cooking oil-II packing is 20% and for 10 litre cooking oil-IV packing, daily removal is 5%. Find

a) Number of daily removed 1 litre cooking oil-II packing.
b) After how many days, 1 litre cooking oil-II are to be replenished?
c) Number of daily removed 10 litre cooking oil-IV packing.
d) After how many days, 10 litre cooking oil-IV packing are to be replenished?

Solution:

Total 1 litre cooking oil-II packing in inventory = 5000

a) Number of daily removed 1lire cooking oil-II packing = 20% of 5000

$= \frac{20}{100} \times 5000$

$= 1000$

b) After 40% removal, replenishment is to be made.

Here 40% of 5000 = 2000

After two days the replenishment is due.

c) Number of daily removed 10 lire cooking oil-IV packing = 5% of 1000

$= \frac{5}{100} \times 1000$

$= 50$

d) Here 40% of 1000 = 400 but packs removed per day are 50.

Therefore $400 \div 50 = 8$

After 8 days the replenishment is due.

EXERCISE 1.3

1. On his last bank statement, Qasim had a balance of Rs. 1,75,000 in his checking account. He wrote one cheque for Rs. 45,790 and another for Rs. 112,921. What is his current balance?
2. Last week Wajid drove 283.4 km on 16.2 litres of petrol. He says that he averaged about 1.75 km/liter. Is his answer reasonable? Explain.
3. Salma bought 3.2 yard of fabric for a total price of Rs. 139.2. How much did the fabric cost per yard?

4. Momina walks 3.5 km/h. She took a 12 h walk. How far did she walk.
5. The hiking club went on a 7day trip. Each day they hiked between 5.5 and 7.5 miles. It is reasonable to assume that clubbing the days the club hiked.
   a. Less than 35 miles
   b. Between 35 and 55 miles
   c. Equally 55 miles
   d. More than 55 miles
6. For a class party the students council purchased 42 balloons at Rs. 1.85 each. What is the total amount the student council paid for the balloons?
7. A group of friends made 4-yard long rectangular banner. They paid Rs. 3.75 per yard for the fabric and Rs.9 for the firm to go around the banner, 10-yard perimeter. What was the width of the banner?
8. A shoe factory has an asset for Rs. 2000,000 of which $\frac{3}{5}$ is the capital and rest is the debt. Find the amount of capital and debt. (Asset = capital + debt)
9. World lowest temperature in past 100 years was recorded to be $-89.2^\circ C$ at Vostok, Antarctica on July 21, 1983. Covert this temperature into Fahrenheit and Kelvin scales.

$$(F = \frac{9}{5}C + 32 \;, \qquad K = ^\circ C + 273)$$

10. A company was penalized by the government act for low quality production. If the company has 3 share holders. Farah, Maryam and Tehreem investing in the ratios of 1 : 2 : 3 and the amount of penalty is Rs. 456,868.97. Find the amount of penalty paid by each of 3 share holders.

KEY POINTS

- Real numbers are union of rational and irrational numbers.
- Basic properties of real numbers are
  - Closure
  - Commutative
  - Associative
  - Identity
  - Inverse
  - Distributive property
- Properties of equality of real numbers:
  - Reflexive
  - Symmetric
  - Transitive
  - Additive
  - Multiplicative
  - Cancellation
- Properties of inequality of real numbers:
  - Trichotomy property
  - Transitive
  - Additive
  - Multiplicative
  - Cancellation
- Positive square roots are called principal square roots.
- For any real numbers a and b and any positive integer n >1 if $a^n = b$, then a is the nth root of b, symbolically it is represented as $a = \sqrt[n]{b}$.

- The symbol $\sqrt{\ }$ is called the radical sign, the number n is called index of the radical and b is called radicand.
- Laws of exponents

(i) $a^m . a^n = a^{m+n} \rightarrow$ Product rule (ii) $\frac{a^m}{a^n} = a^{m-n} \rightarrow$ Quotient rule

(iii) $a^{-n} = \frac{1}{a^n} \rightarrow$ Negative exponent (iv) $(a^m)^n = a^{mn} \rightarrow$ Power to a power rule

(v) $(ab)^n = a^n b^n \rightarrow$ Power of a product rule

(vi) $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \rightarrow$ Power of a quotient rule

MISCELLANEOUS EXERCISE 1

1. Encircle the correct option in the following.

(i) a (b + c – d) equals

(a) a (b + c + d) (b) ac + ab – ad (c) ab + ac + ad (d) ab – ac – ad

(ii) $a^r . a^{-s} \div a^s$ is

(a) a^{r-s} (b) a^{r+2s} (c) $a^r . a^{2s}$ (d) $\frac{a^r}{a^{2s}}$

(iii) $\sqrt[n]{ab}$ is equal to

(a) $\sqrt{ab}$ (b) $n(ab)$ (c) $(ab)^n$ (d) $(ab)^{\frac{1}{n}}$

(iv) Which number is self-multiplicative inverse?

(a) 3 (b) –3 (c) –1 (d) 0

(v) If $a > 0$, then $\sqrt{a}$ is

(a) real (b) integer (c) irrational (d) rational

(vi) If a +b = a, what is value of b?

(a) 1 (b) –1 (c) a (d) 0

(vii) If a.b = 1, what is value of b?

(a) 1 (b) $\frac{1}{b}$ (c) $\frac{1}{a}$ (d) –1

(viii) According to reflexive property : $y^2 - 17 = ?$

(a) $y^2 + 17$ (b) $y - 17$ (c) $y^2 - 17$ (d) $-17 - y^2$

(ix) If a.b = a, what is value of b?

(a) $\frac{1}{a}$ (b) 1 (c) a (d) –1

(x) If a.b = 1, what is b called?

(a) multiplicative inverse of a (b) additive identity

(c) multiplicative identity (d) self-multiplicative inverse

(xi) Commutative property does not hold with respect to:

(a) addition (b) multiplication

(c) subtraction (d) both (a) and (b)

2. Represent each number on the number line.

(i) $-5\frac{1}{5}$ (ii) $\frac{17}{3}$ (iii) $-2 < x < 4$ (iv) $x \geq 6$

3. Write each exponential expression as an equivalent radical expression and simplify if possible.

(i) $(-2)^{\frac{4}{5}}$ (ii) $(-27)^{\frac{1}{3}}$ (iii) $(\sqrt{16})^4$

(iv) $(\sqrt[3]{-8})^9$ (v) $(x^{-2})^3.(x^0)^5$

4. Use the properties of exponents to simplify each of the following.

(i) $\frac{(-2)^3.(-2)^{-4}.(-2)}{(-2)^{-3}}$ (ii) $\frac{2^{\frac{1}{2}}.2^{\frac{3}{4}}}{2^{\frac{1}{2}}} \times \frac{3.3^{\frac{3}{2}}}{3^{-\frac{1}{2}}}$

5. Determine whether each statement is true or false. If false, give an example of a number that shows the statement is ture.

a. Every rational number is an integer.

b. Every real number is an irrational.

c. Every irrational number is a real number

d. Every integer is a rational number.

e. Every real number is either rational number or an irrational number.

UNIT
02

LOGARITHMS

In this unit the students will be able to:

- Express a number in standard form of scientific notation and vice versa.
- Define logarithm of a number to the base *a*.
- Define a common logarithm, characteristic and mantissa of log of a number.
- Use tables to find the log of a number.
- Give concept of antilog and use tables to find the antilog of a number.
- Differentiate between common and natural logarithm.
- Prove the four basic laws of logarithm.
- Apply laws of logarithm to convert lengthy processes of multiplication, division and exponentiation into easier processes of addition and subtraction etc.

October 8, 2005 is an unforgettable day in the history of Pakistan, when the earth started shaking violently and in few minutes the worst disaster had ruined many of the towns and villages from the face of the earth. This earthquake measured 7.6 on Richter scale but what is a Richter scale? Answer to this question will be explained in this unit.

INTRODUCTION

Exponents provide an efficient way of writing very large as well as very small numbers. For example: approximate mass of Uranus is 87 trillion trillion kg i.e. 87 followed by 24 zeros or 87, 000000000000, 000000000000.

This style of expressing a number is called *standard form* which is not useful for such a large number, since some error may occur while writing or telling it. There is another method of writing such numbers, to make them handy.

This method involves integral exponents of 10. In this method the mass of Uranus is $8.7\times10{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000$ kg = 8.7×10^{25} kg. This method is called Scientific notation.

Historya Mystery

Al-Khawarizmi did pioneering work on logarithms and the word **logarithm** is also derived from his name.

2.1 Scientific Notation

A number 'c' is in scientific notation if it is written as $c = d\times10^{n}$, where $1\leq d<10$ and $n\in Z$.

For example: 5.3×10^{7}, 7.412×10^{-2}, 1.592×10^{0}.

How to Write in Scientific Notation

- Place the decimal point after first left hand nonzero digit, the resulting number is d. (Position after first left hand nonzero digit is called **reference position**.)
- Count the number of digits moved by the decimal point. This is absolute value of n.
- If decimal point is moved to left, value of n is positive.
- If decimal point is moved to right, the value of n is negative.

e.g. $0.05\,432 = 05.432\times10^{-2}$ or 5.432×10^{-2}

and $5\,43.2 = 5.432\times10^{2}$

Key Fact

Decimal point is at the right of last digit in an integer.

Example 1: Convert the following into scientific notation:

(a) One light year: 5880,000,000,000 miles

$5\;880000000000.0 = 5.88\times10^{12}$

The number is greater than 10 so exponent must be positive.

(b) Mass of the smallest insect = 0.00000492 g

$0.000004\;92 = 4.92\times10^{-6}$ g

The number is smaller than 1 so exponent must be negative.

Standard Notation: The number already written in scientific notation, can be converted to standard notation by the multiplication of its two factors.

Example 2: Convert the followings into standard notation.

(a) Density of hydrogen = 8.99×10^{-5} g/cm^3

$8.99 \times 10^{-5} = 0\ 00008.99 \times 10^{-5} = 0.0000899$

Exponent is negative so the number is smaller than 1.

Exponent is positive so the number will be greater than 10.

(b) Number of air sacs in lungs = 3.5×10^8

$3.5 \times 10^8 = 3.500000000 \times 10^8 = 350000000.0$ or 350000000

Example 3: The closest star to the Earth (other than Sun) is Alpha Centauri, 4.35 light years from Earth. How many kilometers from Earth is Alpha Centauri?
If one light year = 9460920 million km. Write the answer in scientific notation.

Solution: One light year = 9460920×10^6 km
Distance between Earth and Alpha Centauri
$= 4.35$ light years $= 4.35 \times 9460920 \times 10^6$ km
$= 41155002 \times 10^6 = 4.1155002 \times 10^6 \times 10^7$
$= 4.1155002 \times 10^{13}$ km

Calculator Site

Most of the calculators have a key **E** or **EXP**, For entering a number in scientific notation.

Example 4: The speed of light is approximately 3×10^5 km/s and distance between earth and sun is approximately 1.5×10^8 km. If the sun is suddenly to burn out, how long would it take for earthlings to know about it? Write the answer in standard notation.

Solution: Formula for finding time, if the speed and distance are given, is
Time = Distance/Speed
Here, speed of light = 3×10^5 km/s and distance between Earth and Sun $=1.5\times10^8$km.

$$\text{Time} = \frac{1.5\times10^8\,km}{3\times10^5\,km/s} = 0.5 \times 10^{8-5} \text{ sec} = 0.5 \times {}^{3}10 \text{ sec} = 500 \text{ sec or } 8 \text{ min } 20 \text{ sec}$$

EXERCISE 2.1

1. Write the following in scientific notation.
(i) 0.00053407 (ii) 53400000 (iii) 0.000000000012 (iv) 2.5326

2. Write the following in standard notation.
(i) 9.067×10^{-5} (ii) 5.64×10^0 (iii) 6.53×10^{-6} (iv) 3.1415×10^9

3. Simplify the following by converting into the form indicated.
(i) $563.71 \times 10^{-3} \times 2.54 \times 10^4$ ⟶ scientific notation
(ii) $\dfrac{0.023\times10^5}{10^{-3}}$ ⟶ standard notation
(iii) $\dfrac{2.549\times5067\times10^{-3}}{10^3}$ ⟶ scientific notation
(iv) 0.0009988×10^{10} ⟶ standard notation

4. If it takes 5 seconds to recite 'Kalma Pak' once, how many hours will it take to recite 'Kalma Pak' one million times? Convert hours into days and write the answer in standard form. Round off the answer, discarding the decimal part.
5. Distance between Earth and Sun is 9.3225600×10^7 miles. If speed of light is approximately $1.86{,}000 \times 10^5$ miles per second, how long does it take for light to reach the Earth. Convert the answer in minutes writing in standard form.

2.2 Logarithms

2.2.1 Why We Use Logarithms

Population of the world is growing and the radioactive wastes are decaying continuously. The mathematical tool used to predict the future and explore the past of such rates of growth and decay over time, is an exponential relation.

Key Fact

In late 1500s, John Napier extended the work of Al–Khawarizmi and started developing log tables.

i.e. an equation of the form $x = b^y$ where b, x and y are real numbers, $b > 0$, $x > 0$ and $b \neq 1$. This relation is widely used by archaeologists, scientists and business people. The inverse relation of this exponential relation is called logarithmic relation.

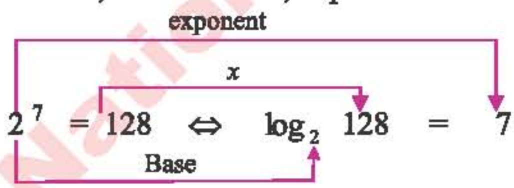
Definition of Logarithm

If $b^y = x$ where $x, y, b \in R$; $b > 0, x > 0$ and $b \neq 1$, then y is the **logarithm** of x with base b, written as $y = \log_b x \Leftrightarrow b^y = x$.

While evaluating logarithms, remember that a logarithm is an exponent, e.g. if $\log_9 81 = 2$, then 2 is the logarithm of 81 with base 9, since 9 raised to power 2 gives 81.

Example 5: Convert the following exponential equations to logarithmic equations and the logarithmic equations to exponential equations.

(a) $2^7 = 128$, here base = 2, exponent = 7 and $x = 128$



Enlighten Yourself

Exponential equations are used by
a. Archaeologists, for finding the age of very old bones, fossils etc.
b. Scientists for finding the life time of radioactive elements etc.

(b) $7^{-3} = \dfrac{1}{343} \Leftrightarrow \log_7 \dfrac{1}{343} = -3$

(c) $\sqrt[3]{125} = 5$ or $(125)^{\frac{1}{3}} = 5$

$125^{\frac{1}{3}} = 5 \Leftrightarrow \log_{125} 5 = \dfrac{1}{3}$

(d) $\log_5 625 = 4 \quad \Leftrightarrow \quad 5^4 = 625$

(e) $\log_2 \frac{1}{64} = -6 \quad \Leftrightarrow \quad 2^{-6} = \frac{1}{64}$

(f) $\log_{81} \frac{1}{3} = -\frac{1}{4} \quad \Leftrightarrow \quad (81)^{-\frac{1}{4}} = \frac{1}{3}$

Check Point

$\text{Log}_t 5^0 = ?$
$\text{Log}_3 (\log_2 2) = ?$

Key Fact

- $\log_b x$ is defined only for positive x.
- $\log_b 1 = 0 \quad \because b^0 = 1$
- $\log_b b = 1 \quad \because b^1 = b$
- $\log_a a^x = x \quad \because a^x = a^x$
- $\log_b x_1 = \log_b x_2 \Rightarrow x_1 = x_2$

Example 6: Check whether these logs are defined or not?

(a) $\log_1 2 = y \Rightarrow 1^y = 2$

None of the exponents of 1 can give answer 2, so $\log_1 2$ is undefined.

(b) $\log_5 (-1) = y \Rightarrow 5^y = -1$

None of the exponents of 5 can give answer '–1', so $\log_5 (-1)$ is undefined i.e. log of negative number is not defined.

(c) $\log_2 0 = y \Rightarrow 2^y = 0$

None of the exponent of 2, can give answer 0, so log of 0 is not defined.

(d) Is $\log_2 (4 - 2x) = y$ true or not for $x = 0, 1, 2$.

$\log_2 (4 - 2x) = y \Rightarrow 2^y = 4 - 2x$

If $x = 0$, then $2^y = 4$ is true for $y = 2$.

If $x = 1$, then $2^y = 2$ is true for $y = 1$.

If $x = 2$, then $2^y = 0$ is not true for any value of y.

Example 7: Find the value of unknowns by converting logarithmic form to exponential form.

(a) $\log_2 x = 4 \Rightarrow 2^4 = x \Rightarrow x = 2 \times 2 \times 2 \times 2 = 16$

(b) $\log_{64} x = -\frac{4}{3} \Rightarrow (64)^{-\frac{4}{3}} = x \Rightarrow x = (4^3)^{-\frac{4}{3}} = 4^{3 \times -\frac{4}{3}} = \frac{1}{4^4} = \frac{1}{256}$

(c) $\log_b \frac{1}{128} = -7 \Rightarrow b^{-7} = \frac{1}{128} = \frac{1}{2^7}$ or $b^{-7} = 2^{-7}$

As exponents are same, so bases must be same i.e $b = 2$

(d) $\log_{27} 3 = y \Rightarrow 27^y = 3$ or $(3^3)^y = 3 \Rightarrow 3^{3y} = 3^1$

As bases are same so exponents must be same. i.e. $3y = 1$ or $y = \frac{1}{3}$

Check Point

Decide which log is defined:

Log 1	Log 0
Log–2	$\text{Log}_{-10} 2$

Example 8: Find y if $\log_b (y^3 + 1) = \log_b 28$

Solution: $\log_b (y^3 + 1) = \log_b 28$

$\Rightarrow y^3 + 1 = 28$ or $y^3 = 28 - 1 = 27$

$y = \sqrt[3]{27} = 3$

EXERCISE 2.2

1. Check whether $\log_x(7-x)$ is defined for
   (i) $x=0$ (ii) $x=1$ (iii) $x=6$ (iv) $x\geq 7$
2. Convert the form of following equations i.e. from exponential form to logarithmic form and vice versa
   (i) $\log_6 216=3$ (ii) $7^4=2401$ (iii) $\log_5 x=5$ (iv) $b^{-\frac{3}{4}}=\frac{1}{27}$
   (v) $125^{\frac{x}{3}}=25$ (vi) $\log_{10}10^{12}=y$ (vii) $(256)^{\frac{x}{4}}=\frac{1}{64}$
   (viii) $\log_3(x^3+1)=2$ (ix) $\log_5(2x-3)=1$ (x) $2x+1=2^3$
3. Find the value of x in the following questions.
   (i) $\log_x 3=1$ (ii) $\log_{x+1}9=2$ (iii) $\log_3 81=x$
   (iv) $\log_2 64=x+1$ (v) $\log_2 x=4$ (vi) $\log_2(x^2-1)=3$
4. Find the unknowns appeared in the question 2.

2.2.2 Common Logarithm

There are two most commonly used bases for logarithm i.e.'10' and 'e $\approx$2.71828'(an irrational number). Base 10 was used by Henry Briggs.

If $10^y=x$, for $x>0$, then y is called ***common log*** *of x i.e. $10^y=x \Leftrightarrow \log_{10}x=y$*

These logarithms are also called Briggs's logarithms, denoted by $\log_{10}x$ or simply $\log x$. If none of the base is mentioned with log then it is obviously a common logarithm e.g. $\log_{10}36$ can be simply written as log 36.

Logarithm of a number = Characteristic + Mantissa

2.2.3 Characteristic

Integral part of the logarithm is called ***characteristic.***

Characteristic is an integer. It is infact the integral power of 10, when the number is written in scientific notation. e.g characteristic of $\log 3.3\times 10^2$ is '2' and in $\log 5.632\times 10^{-4}$, characteristic is negative 4. This negative characteristic is usually written as $\bar{4}$.

Characteristic of the log of some number can also be found using **reference position**. In 0.00532, the reference position is between 5 and 3. By counting the number of digits between the decimal point and the reference position we get the numerical value of the characteristic however, the sign is taken negative if the reference position is on right side of the decimal point and it is taken positive otherwise.

Memory Plus

If $\log x_1=\log x_2 \Rightarrow x_1=x_2$
- $\log 10^0=0$
  $\log 1=0$
- $\log 10^1=1$
- $\log 10^2=2$
- $\log 10^3=3$
- $\log 10^{12}=12$

So log of an integral power of 10 is that integral whole number,
- lso If $1<x<10$,
  then $0<\log x<1$.

2.2.4 Mantissa

Decimal part or the fractional part of a logarithm is called mantissa.

Mantissa is always a nonnegative number less than 1, i.e. it can be either zero or positive but never negative. Mantissa is found from the log table.

A small part of log table:

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170						4	9	13	17	21	26	30	34	38
						0212	0253	0294	0334	0374	4	8	12	16	20	24	28	32	36
11	0414	0453	0492	0531	0569						4	8	12	15	19	23	27	31	35
						0607	0645	0682	0719	0755	4	7	11	15	19	22	26	30	33
12	0792	0828	0864	0899	0934						3	7	11	14	18	21	25	28	32
						0969	1004	1038	1072	1106	3	7	10	14	17	20	24	27	31
13	1139	1173	1206	1239	1271						3	7	10	13	16	20	23	26	30
						1303	1335	1367	1399	1430	3	7	10	13	16	19	22	25	29
14	1461	1492	1523	1553	1584						3	6	9	12	15	19	22	25	28
						1614	1644	1673	1703	1732	3	6	9	12	15	17	20	23	26
15	1761	1790	1818	1847	1875						3	6	9	11	14	16	20	23	26
						1903	1931	1959	1987	2014	3	6	8	11	14	17	19	22	24

Example 9: Find **(a)** log 156.3 **(b)** log 0.0123

Solution: (a) $\log 156.3 = \log 1.563 \times 10^2$

characteristic = 2

Convert the number into scientific notation

Explanation: Llook at the log table in the extreme left column for the number 15. The next digit in 156.3 is 6 From the top row, look at the digit 6. Move vertically downward from 6 and horizontally rightwards from 15. The number present at the intersection of row of 15 and the column of 6 is 1931. Go ahead horizontally and see the number present at the intersection of row of 15 and column of 3 (in the difference tables) i.e. 8. Add 1931 and 8 to get 1939. Since mantissa is less than 1, so mark the decimal point before first digit so mantissa is '.1939'.

mantissa = .1939 or 0.1939

log 156.3 = characteristics + mantissa = 2 + 0.1939 = 2.1939

(b) $\log 0.0123 = \log 1.230 \times 10^{-2}$

characteristic = -2 or $\bar{2}$

For mantissa, use the log table to see the number present at the intersection of row of 12 and the column of 3 i.e. 0899, as there is no difference table for '0' so mark the decimal point before the first digit i.e. mantissa is .0899.

mantissa = .0899

log 0.0123 = $\bar{2}$ + .0899 = $\bar{2}$.0899 (never write '–2.0899')

Example 10: Find log 1009

Solution:

$\log 1009 = \log 1.009 \times 10^3$

characteristic = 3

mantissa = .0038 ($\neq$ .38)

$\therefore$ log 1009 = 3.0038

10 | 0 | 9

.0000
+.0038
.0038

Memory Plus

Number of digits in a whole number = characteristics +1

i.e If characteristic of log of some whole number is 3, then the number of digits in that number will be 3 +1=4

see Example 10 for confirmation.

EXERCISE 2.3

Find the logarithms of following numbers if possible.

1. 5313	2. 4580	3. 9.613	4. 110.9
5. 52.39	6. 0.01207	7. 0.0093	8. 1 Trillion
9. 0 .00004	10. 4	11. 4000	12. 54
13. 1.009	14. 0.1009	15. – 4	

2.2.5 Antilogarithm

The inverse operation of taking log is called antilog. An antilog is used to cancel the effect of log.

If $\log x = y$ *then* x *is called the **antilog** of* y or $x = \text{antilog } y$.

Antilogarithm tables are used for finding it. Before finding antilog, recall that log x consists of two parts, characteristic and mantissa. For finding antilog, mantissa is used for looking in the antilog table. Characteristic is not used in table, however characteristic is used for locating the decimal point in the number obtained from antilog table.

A small part of Antilog of Table:

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
.45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
.46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
.47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
.48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
.49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

Example 11(a): Find antilog 2.4541

Solution: 2.4541 is obviously log of some number so,

characteristic = 2 (integral part) , mantissa = .4541 (fractional part)

Explanation: Look at the antilog table for .45 in extreme left column and 4 (the digit next to .45) in the top row. The number present at the intersection of row of .45 and column of 4 is 2844. Now go ahead horizontally, the number present at the intersection of row of .45 and difference column of 1 is 1. Add 2844 and 1 to get 2845 . Antilog table is no more needed, however the antilog of 3.4541 is not yet completely found. Locate the reference position in the number obtained from antilog table i.e. $2 \wedge 845$, now characteristic will locate the decimal point . As characteristic is +2, so mark the decimal point moving two digits rightwards from reference position i.e. $2 \wedge 84 . 5$

$\therefore$ antilog 2.4541 = 284.5

(b) If $\log x = \bar{2}.0000$, then find x.

Solution: $\log x = \bar{2}.0000$

$\text{antilog}(\log x) = \text{antilog}(\bar{2}.0000)$ (taking antilog on both sides.)

$x = \text{antilog}(\bar{2}.0000)$

here, characteristic = $\bar{2}$ and mantissa = .0000 (see log table at the back of this book.)

$\therefore\ x = \text{antilog}\ (\bar{2}.0000)$
$\quad = .01000$

Example 12: Find antilog of $\bar{2}.4900$

Solution: antilog $\bar{2}.4900$

characteristic = $\bar{2}$ mantissa = .4900

antilog $(\bar{2}.49) =$ .03∧090 = 0.03090

Rough work for antilog

.49 0 0 → 3090

EXERCISE 2.4

Find the antilog of following numbers.

1.	2.4324	2.	1.5890	3.	0.2425	4.	3.5636
5.	4	6.	0.0038	7.	$\bar{1}.2429$	8.	$\bar{2}.9281$
9.	$\bar{3}.5219$	10.	0.0000	11.	–3	12.	5.9990
13.	2.4900	14.	0.49000	15.	2.34		

2.3 Common and Natural Logarithm

John Napier started developing log tables with base *e*, so the logarithms with base *e* are called natural logarithms or Naperian logarithms, represented by '*ln x*'.

2.3.1

*If $e^y = x$, for positive values of x then y is called **natural log** of x i.e. $e^y = x \Leftrightarrow y = \ln x$.*

Napier spent last 20 years of his life working with log tables of base *e*, which he never finished and died. Henry Briggs, then completed these tables. The difference between common and natural logarithms is depicted below.

Log Type	Representation	Base	Nature of base	Properties
Common (Briggs)	$\log x$	10	Rational	$\log 1 = 0$ $\log 10 = 1$ $\log 10^x = x$
Natural (Naperian)	$\ln x$	$e \approx 2.71828$	Irrational	$\ln 1 = 0$ $\ln e = 1$ $\ln e^x = x$

2.4 Laws of Logarithms

Laws of logarithms are closely related to the laws of exponents, since logarithms in nature, are exponents. The laws of exponents are given in unit 1. In this section the laws of exponents are used to develop some laws of logarithms, for solving complicated equations involving exponents and logarithms. Also the complicated questions of multiplication, division and root extraction are converted to handy sums of addition and subtraction.

2.4.1 Product Law of Logarithms

For any real numbers m, n and b where $b \neq 1$, $\log_b mn = \log_b m + \log_b n$

Proof: Let $\log_b m = x$ .... (i) and $\log_b n = y$ ...... (ii)

Their respective exponential equations are

$m = b^x$ ...... (iii) and $n = b^y$ ...... (iv)

Now product of equations (iii) and (iv) is

$m \times n = b^x . b^y$

$mn = b^{x+y}$ ← using product rule of exponents

$\Rightarrow \log_b mn = x + y$ ←logarithmic form of above equation

or $\log_b mn = \log_b m + \log_b n$ ← substituting the values of x and y

The Product law of logarithms states that:

Logarithm of a product of two (or more) numbers is equal to the sum of their logarithms, provided that all logarithms are defined.

Key Fact

$\log_{10}(3 \times 2) = \log_{10}3 + \log_{10}2$

Example 13 (a): Use the product rule to expand $\log_e(x^2 y^3 z)$.

Solution $\log_e(x^2 y^3 z) = \log_e x^2 + \log_e y^3 + \log_e z$

(b) Use product rule to combine $ln\, a + ln\sqrt{b} + ln\, c^3$ in a single logarithmic term.

Solution $ln\, a + ln\sqrt{b} + ln\, c^3 = ln\,(a\sqrt{b}\, c^3)$

2.4.2 The Quotient Law of Logarithm

For any positive numbers m, n and b, where $b \neq 1$.

$\log_b \frac{m}{n} = \log_b m - \log_b n$

Proof: Let $\log_b m = x$ ...... (i) and $\log_b n = y$...... (ii)

Respective exponential forms of above equations are

$m = b^x$ ...... (iii) and $n = b^y$ ...... (iv)

Dividing (iii) by (iv) i.e. $\frac{m}{n} = \frac{b^x}{b^y}$

$\frac{m}{n} = b^{x-y}$ ←quotient rule of exponents with same base

$\Rightarrow \log_b \frac{m}{n} = x - y$ ←conversion to logarithmic form

$\log_b \frac{m}{n} = \log_b m - \log_b n$ ←Substituting the value of x and y

The Quotient law of logarithms states that:

Logarithms of quotient of two numbers is equal to the difference of their logarithms provided that all the logarithms are defined.

Example 14 (a): Use quotient law of logarithms to expand $\log \frac{13}{7b}$.

(b) Use quotient law of logarithms to combine log 39 – log t – log b into single logarithmic term.

Solution (a): $\log \frac{13}{7b} = \log 13 - (\log 7 + \log b) = \log 13 - \log 7 - \log b$

Solution (b): $\log 39 - \log t - \log b = \log 39 - (\log t + \log b) = \log 39 - \log tb = \log \frac{39}{tb}$

Key Fact

$\log(50-5) \neq \log 50 - \log 5$

2.4.3 The Power Law of Logarithm

For any real number m, n and b, where $m > 0$, $b > 0$, $b \neq 1$, $\log_b m^n = n \log_b m$

Proof: Let $\log_b m = x$

Converting into exponential equation i.e $m = b^x$

$m^n = (b^x)^n$ ← taking n^{th} power on both sides.

or $m^n = b^{nx}$ ← using power of power rule .

$\Rightarrow \log_b m^n = nx$ ← converting into logarithmic form .

$\log_b m^n = n \log_b m$ ← substituting the value of x.

The Power Law of logarithms states that:

Logarithm of power of a number is equal to the power times the logarithm of the number, provided that all logarithms are defined.

Example 15(a): Use power law of logarithms to expand $\log_2 100^{-3}$.

Solution: $\log_2 100^{-3} = -3 \log_2 100$

(b) Use power law of logarithms to combine $-\frac{4}{3} \log_{\sqrt{3}} 7$

Solution: $-\frac{4}{3} \log_{\sqrt{3}} 7 = \log_{\sqrt{3}} 7^{-\frac{4}{3}}$

2.4.4 Change of Base Law of Logarithms

Although only common logarithms and natural logarithm are programmed into a calculator still the logarithms for other positive real bases can be found by changing that base into some frequently used base that is 'e' or '10'. The law which enables us to change the base is called change of base law which is given below.

If a, b and m are positive real numbers and, $a \neq 1$, $b \neq 1$, then

$\log_a m = \log_b m \cdot \log_a b$

Proof: Let $\log_b m = x$

$\Rightarrow \quad m = b^x$ ←converting above equation into exponential equation.

$\log_a m = \log_a b^x$ ←taking log on both sides with base 'a'.

$\log_a m = x.\log_a b$ ←applying power law of logarithm.

$\log_a m = \log_b m.\log_a b$ or $\log_b m = \dfrac{\log_a m}{\log_a b}$

Slide Rule based upon laws of logarithms were used for complicated calculations, before the invention of calculators.

Example 16: Convert the base of $\log_b 536$ into 10.

Solution: $\log_b 536 = \dfrac{\log_{10} 536}{\log_{10} b}$ or $\dfrac{\log 536}{\log b}$

Key Fact

The change of base law and the quotient law of logarithms are often confused. Remember the difference between $\log_b(\dfrac{m}{n})$ and $\dfrac{\log_a m}{\log_a n}$. These two expressions look alike, however they are totally different.

Example 17(a): Use laws of logarithms to expand $\log \dfrac{5p^2q^{\frac{1}{2}}}{4\sqrt{st^3}}$.

Solution: $\log \dfrac{5p^2q^{\frac{1}{2}}}{4\sqrt{st^3}} = \log(5p^2q^{\frac{1}{2}}) - \log(4\sqrt{s}\ t^3)$ ←using quotient law of log

$= \log 5 + \log p^2 + \log q^{\frac{1}{2}} - [\log 4 + \log\sqrt{s} + \log t^3]$ ←using product law of log

$= \log 5 + 2\log p + \dfrac{1}{2}\log q - \log 4 - \dfrac{1}{2}\log s - 3\log t$ ←using power law of log

(b) Use laws of logarithms to evaluate

(i) $\log_2 5\sqrt{3}$ (ii) $\log_2\left(\log_2 8^2 - \log_{\sqrt{3}} 27 + \log_{\sqrt{10}} 10\right)$

Solution: (i) $\log_2 5\sqrt{3} = \dfrac{\log_{10} 5\sqrt{3}}{\log_{10} 2}$ ← change of base law

$= \dfrac{\left[\log 5 + \log 3^{\frac{1}{2}}\right]}{\log 2} = (\log 5 + \dfrac{1}{2}\log 3) \div \log 2$

$= [0.6990 + \dfrac{1}{2}(0.4771)] \div (0.3010)$

$= \dfrac{0.9376}{3.010} = 3.1148$

(ii) Evaluate $\log_2\left(\log_2 8^2 - \log_{\sqrt{3}} 27 + \log_{\sqrt{10}} 10\right)$

$\log_2\left(\log_2 8^2 - \log_{\sqrt{3}} 27 + \log_{\sqrt{10}} 10\right)$

$$= \log_2\left(\log_2 (2^3)^2 - \log_{\sqrt{3}}\left(\left(\sqrt{3}\right)^2\right)^3 + \log_{\sqrt{10}}\left(\sqrt{10}\right)^2\right)$$

$$= \log_2\left(\log_2 2^6 - \log_{\sqrt{3}}\left(\sqrt{3}\right)^6 + \log_{\sqrt{10}}\left(\sqrt{10}\right)^2\right)$$

$$= \log_2\left(6 \log_2 2 - 6 \log_{\sqrt{3}} \sqrt{3} + 2 \log_{\sqrt{10}} \sqrt{10}\right) \qquad (\because \log_b b^x = x)$$

$$= \log_2(6 - 6 + 2)$$

$$= \log_2 2 = 1 \qquad (\because \log_b b = 1)$$

c. If $\log_b 2 = 0.3010$, $\log_b 3 = 0.4771$ and $\log_b 5 = 0.6990$, then evaluate $\log_b 0.0036$, applying laws of logarithms.

> **Math Play Ground**
>
> Maths play ground
> 1. Take students to the play ground.
> 2. Give each student a strip of paper with a simple logarithms sum written on it.
> 3. Spread answers of all questions in the play ground and ask students to find their respective answers.
> 4. Specimen questions may include:
> (i) log10 = ? (ii) lne = ? etc.

Solution: $\log_b 0.0036 = \log_b \left(\dfrac{36}{10000}\right)$

$$= \log_b \left(\frac{2^2 \times 3^2}{2^4 \times 5^4}\right)$$

$$= \log_b \left(\frac{3^2}{2^2 \times 5^4}\right)$$

$$= 2 \log_b 3 - 2 \log_b 2 - 4 \log_b 5$$

$$= 2(0.4771) - 2(0.3010) - 4(0.6990)$$

$$= -2.4438$$

(which is negative number having both characteristic and mantissa as negative.)
Since mantissa can never be negative, to make the mantissa positive check the next positive integer to the magnitude of the answer. The next integer is '+3' as shown on the number line

Now 'add 3 to' and 'subtract 3 from' the answer
i.e. $\log_b 0.0036 = -2.4438 + 3 - 3 = 0.5562 - 3$.
The positive term is mantissa and negative term is characteristic.
So, $\log_b 0.0036 = \bar{3}.5562$

Important results deduced from Laws of Logarithms

(i) $\log_b a \times \log_a b = 1$	(ii) $\log_c a = \dfrac{1}{\log_a c}$
(iii) $\log_b a \, . \, \log_c b = \log_c a$	(iv) $\log_s r^t \times \log_t s^r \times \log_r t^s = rst$
(v) $\log_x z \times \log_y x \times \dfrac{1}{\log_y z} = 1$	(vi) $\log_b a \, . \, \log_c b \times \log_a c = 1$

EXERCISE 2.5

1. Use laws of logarithms to expand the followings.

(i) $\log 9t$ **(ii)** $\log \frac{59}{s}$ **(iii)** $\log \frac{(5pq^2)}{(xy^3)}$ **(iv)** $\log \sqrt{\frac{53.3}{46.4}}$

(v) $\log \left(\frac{5^2 t^5 a^{\frac{1}{3}}}{\sqrt[3]{4.4tb^3}}\right)$ **(vi)** $\log \sqrt[5]{\frac{7^2 t^3 p}{d^6 b^2}}$ **(vii)** $\log \left(\frac{\sqrt[3]{5.512}pm^{\frac{1}{2}}}{\sqrt[4]{5.91}a^2 b}\right)^t$

2. Use laws of logarithms to combine the followings into single logarithmic terms.

(i) $3\log x - 5\log y$

(ii) $\frac{1}{2}\log t + \frac{1}{3}\log r - \frac{1}{5}\log s$

(iii) $\frac{1}{7}[\log 57.7 - 3\log 9.24 + 4\log 36.6 - 2\log 23.3]$

(iv) $5\log 6 - 7\log 9.42 + \frac{1}{3}\log t - \frac{1}{2}\log 32.2 + \frac{2}{3}\log a$

(v) $\frac{5}{4}\log 37.74 - \frac{1}{4}\log 53.71 + \frac{1}{4}\log 28.83$

3. Use laws of logarithms to evaluate the followings.

(i) $\log_2 15$ **(ii)** $\log_9 \sqrt[3]{9}$ **(iii)** $\log_3 65$ **(iv)** $\log_{\sqrt{3}} 72.34$ **(v)** $\log_{\sqrt{7}} 343$

4. If $\log_b 2 = 0.3010$, $\log_b 3 = 0.4771$, $\log_b 5 = 0.6990$
then evaluate the followings with laws of logarithms.

(i) $\log_b \frac{6}{5}$ **(ii)** $\log_b \frac{100}{9}$ **(iii)** $\log_b \frac{\sqrt[3]{450}}{\sqrt{27}}$ **(iv)** $\log_b 0.024$ **(v)** $\log_b \sqrt[7]{5\frac{2}{5}}$

2.5 Applications of Logarithms

Common logarithms appear in many scientific formulae.

Richter Scale

Opening of this unit depicts the 2005 earthquake in Pakistan, when thousands of people lost their lives. The Richter Scale used for measuring the magnitude (M) of earthquake is a logarithmic scale. If I is intensity of its shock waves and I_0 is a constant then

$$M = \log\left(\frac{I}{I_0}\right)$$

Example 18: An earthquake that occurred in Pakistan in 2005, measured 7.6 on the Richter scale. In 1978, an earthquake in China measured 8.2 on the Richter scale. How many times, the China's earthquake was stronger than the Pakistan's earthquake.

Solution: Let I_1 be the intensity of Pakistan's earthquake and I_2 be the intensity of China's earthquake.

As, $M = \log\left(\frac{I}{I_0}\right)$

Then, $7.6 = \log\left(\frac{I_1}{I_0}\right) = \log I_1 - \log I_0$ .... (i)

$8.2 = \log\left(\frac{I_2}{I_0}\right) = \log I_2 - \log I_0$ .... (ii)

> **Project**
>
> Search more applications of logarithms in Biology, Chemistry and Physics.

Subtracting (i) from (ii)

$\log I_2 - \log I_1 = 8.2 - 7.6$

$\Rightarrow \quad \log\left(\frac{I_2}{I_1}\right) = 0.6$

$\text{antilog}\log\left(\frac{I_2}{I_1}\right) = \text{antilog } 0.6$ (taking antilog on both sides)

$\frac{I_2}{I_1} = 3.98 \approx 4$

$\therefore$ China's earthquake is nearly 4 times stronger than Pakistan's earthquake.

Example 19: Use laws of logarithms to evaluate.

(a) $\frac{\sqrt[3]{8.59} \times (55.6)^2}{2.51 \times \sqrt{2.12}}$ **(b)** $\sqrt[3]{5\frac{1}{3}} \div \sqrt{\frac{22}{7}}$

Solution: **(a)** let $x = \frac{\sqrt[3]{8.59} \times (55.6)^2}{2.51 \times \sqrt{2.12}}$

Then, by taking common log on both sides.

$$\log x = \log \frac{\sqrt[3]{8.59} \times (55.6)^2}{2.51 \times \sqrt{2.12}}$$

$$= \log(8.59)^{\frac{1}{3}} + \log(55.6)^2 - \log(2.51) - \log(2.12)^{\frac{1}{2}}$$

$$= \frac{1}{3}\log 8.59 + 2\log 55.6 - \log 2.51 - \frac{1}{2}\log(2.12)$$

$$\approx \frac{1}{3}(0.9340) + 2(1.7451) - (0.3997) - \frac{1}{2}(0.3263)$$

$$\approx 0.3113 + 3.4902 - 0.3997 - 0.1632$$

$\log x \approx 3.2386$

$\text{antilog}\log x \approx \text{antilog } 3.2386$ ← Taking antilog on both sides

$x \approx 1732.21$

$\therefore \frac{\sqrt[3]{8.59} \times (55.6)^2}{2.51 \times \sqrt{2.12}} \approx 1732.21$

(b) let $x = \sqrt[3]{5\frac{1}{3}} \div \sqrt{\frac{22}{7}}$

$\log x = \log \sqrt[3]{5\frac{1}{3}} \div \sqrt{\frac{22}{7}}$ ← Taking common log on both sides

$\log x = \log \left(\frac{16}{3}\right)^{\frac{1}{3}} - \log \left(\frac{22}{7}\right)^{\frac{1}{2}} = \frac{1}{3}\log\left(\frac{16}{3}\right) - \frac{1}{2}\log\left(\frac{22}{7}\right)$

$= \frac{1}{3}(\log 16 - \log 3) - \frac{1}{2}(\log 22 - \log 7)$

$= \frac{1}{3}\log 16 - \frac{1}{3}\log 3 - \frac{1}{2}\log 22 + \frac{1}{2}\log 7$

$\approx \frac{1}{3}(1.2041) - \frac{1}{3}(0.4771) - \frac{1}{2}(1.3424) + \frac{1}{2}(0.8451)$

$\log x \approx 0.4014 - 0.1590 - 0.6712 + 0.4226$

$\log x \approx -0.0062$

$\log x \approx (-0.0062 + 1) - 1$ ← making mantissa positive

$\approx 0.9938 - 1$

$\log x \approx \bar{1}.9938$

Taking antilog on both sides.

antilog $\log x \approx$ antilog $\bar{1}.9938$

$x \approx 0.9858$

Example 20: Find the number of digits in 5^{50}.

Solution:

By finding the log of the given whole number, we can find the relevant characteristic. But the number of digits in a whole number is always one more than the characteristic of the log of that number.

Now $\log 5^{50} = 50 \log 5$

$= 50 \times (.6990) = 34.95$

Characteristic = 34

Number of digits in 5^{50} = characteristic $+ 1 = 34 + 1 = 35$

EXERCISE 2.6

1. Find the number of digits in
   (i) 3^{30} (ii) 100^{100} (iii) 2^{10} (iv) 5^{37} (v) 529^{30} (vi) 23^{15}
2. Evaluate applying laws of logarithms.
   (i) 23.57×5.967 (ii) $\frac{65.89}{7.392}$ (iii) $\frac{47.27 \times 5.321}{9.712 \times 4.171}$ (iv) $\frac{\sqrt[3]{27.98}}{\sqrt[2]{28.73}}$
   (v) $\frac{\sqrt[7]{129.4}}{\sqrt[3]{27.37}}$ (vi) $\frac{\sqrt{39.24} \times \sqrt[3]{1.931}}{\sqrt[4]{64.4} \times \sqrt{23.91}}$ (vii) $\frac{\sqrt{16\frac{3}{4}}}{\sqrt[3]{53}}$
   (viii) $\frac{(27.98)^2}{(28.73)^3}$
3. The Kansu, China earthquake of 1920 was measured about 8.5 on Richter Scale and the Tokyo, Japan earthquake of 1923 was measured 7.8 on that scale how many times stronger was the 1920 earthquake than 1923 earthquake?

KEY POINTS

- A number written as $d \times 10^n$,(where $1 \le d < 10$, $n \in Z$) is said to be in Scientific notation.
- Reference position is the place after first left hand non–zero digit.
- For $x, y, b \in R$; $b > 0, x > 0$ and $b \ne 1$, y is called logarithm of x with base b written as
  $$y = \log_b x$$
- Logarithm of a number consists of two parts,
  - Characteristic: The integral part of logarithm.
  - Mantissa: The fractional part of logarithm, which is never negative.
- If $\log x = y$ then x is called the antilog of y.
- $y = \log_{10} x$, is called common logarithm and $y = \log_e x$ is called natural logarithm.
- $\log_b mn = \log_b m + \log_b n$
- $\log_b \frac{m}{n} = \log_b m - \log_b n$
- $\log_b m^n = n \log_b m$
- $\log_n m = \frac{\log_b m}{\log_b n}$

MISCELLANEOUS EXERCISE 2

1. Encircle the correct option in the following.

(i) If $a = b \times 10^n$ is written in scientific notation then ,

(a) $0 \le b \le 10$ **(b)** $0 \le b < 10$ **(c)** $1 \le b \le 10$ **(d)** $1 \le b < 10$

(ii) In 0.537, reference position is

(a) after 0 **(b)** after 7 **(c)** after 5 **(d)** before 7

(iii) $\log_1 100$ is

(a) 2 **(b)** –2 **(c)** 0 **(d)** impossible

(iv) If $\log(x+3) = \log(15x-4)$ then x is

(a) 0.5 **(b)** 7 **(c)** 14 **(d)** 2

(v) $\log_7 7^{-3} + \log_2 4^3$ is

(a) 3 **(b)** –3 **(c)** 0 **(d)** ± 3

(vi) For the log 0.00327, characteristic is

(a) –2 **(b)** –3 **(c)** 3 **(d)** 0

(vii) $\log_b(M+N)$ is

(a) $\log_b MN$ **(b)** $\log_b M + \log_b N$ **(c)** both a and b **(d)** none of these

(viii) $\log_b g^h$ is

(a) $g \log_b h$ **(b)** $\log_b(gh)$ **(c)** $(\log_b g) \times h$ **(d)** $h \log_g b$

(ix) $\log_b M - \log_b N$ is

(a) $\dfrac{\log_b M}{\log_b N}$ **(b)** $\log_b \dfrac{M}{N}$ **(c)** $\log_N M$ **(d)** $\dfrac{\log_b N}{\log_b M}$

(x) $\log_{\sqrt{10}} 100^2$ is

(a) 2 **(b)** 1 **(c)** 4 **(d)** 8

(xi) log18 is

(a) 3 log2 + log3 **(b)** log 2 + 2log3 **(c)** 3log 3 + 2log2 **(d)** 2 log 3 + 3 log 2

(xii) log 5 – log 8 + log 3 – log 2 is

(a) $\log\dfrac{5\times 2}{8\times 3}$ **(b)** $\log\dfrac{15}{16}$ **(c)** $\log\dfrac{30}{8}$ **(d)** log –2

(xiii) $\log_{10} 100^0$ is

(a) 2 **(b)** 0 **(c)** 1 **(d)** impossible

(xiv) Scientific notation of 6.25 is

(a) 6.25×10^1 **(b)** 6.25×10^0 **(c)** 6.25×10 **(d)** 0.625×10^2

(xv) Base of natural logarithm is

(a) rational number **(b)** integer **(c)** irrational number **(d)** 10

(xvi) If $\log_{\sqrt{x}} 25 = 4$ then x is

(a) $+5$ **(b)** -5 **(c)** ±5 **(d)** impossible

(xvii) $\log_{\sqrt{b}} 10^4 \div \log_{\sqrt{b}} 10$ is

(a) $\log_{\sqrt{b}} \frac{10^4}{10}$ **(b)** $\log_{\sqrt{b}} 10^4 - \log_{\sqrt{b}} 10$ **(c)** 4 **(d)** $\log_{\sqrt{b}} (10^4 - 10)$

(xviii) $5 \log 2 - 2 \log 5$ is

(a) $\frac{(\log 2)^5}{(\log 5)^2}$ **(b)** $\frac{\log 2^5}{\log 5^2}$ **(c)** $\log \frac{2^5}{5^2}$ **(d)** $\frac{5}{2}\log\frac{2}{5}$

2. Convert the following into scientific notation.

(i) 53.36 (ii) 0.0000000000000102 (iii) 523.4×10^{-3}

3. Convert the following into standard notation.

(i) 7.232×10^{-2} (ii) $10.53 \times 10^2 \times 20.31$ (iii) 5.6×10^0

4. Evaluate the following.

(i) $\log_5 5^3 - \log_2 2^3$ (ii) $\log_2 4 - \log_3 1$ (iii) $\log_8(\log_x x - \log_b b^{-7})$

5. Find x if

(i) $\log_3 9 - \log_b 1 = x$ (ii) $\log_2 x - \log_2 16^{1/4} = 3$

(iii) $\log_2 (x^2 - 1) = \log_2 3$ (iv) $\log_7 x = \log_7 (8 \log_y y)$

6. If $\log_b 2 = 0.3010, \log_b 3 = 0.4771$ & $\log_b 5 = 0.6990$, then evaluate the following by applying laws of logarithms.

(i) $\log_b 30$ (ii) $\log_b 0.24$ (iii) $\log_b 360$

7. Simplify with the help of laws of logarithm $\left(\frac{0.5327\times\sqrt[3]{42.97}}{0.0059}\right)^3$.

8. Prove that:

$$\log_v U \times \log_w V \times \log_u W = 1$$

UNIT
03

SETS AND RELATIONS

In this unit the students will be able to:

- Describe mathematics as a study of patterns, structures and their relationships.
- Identify sets and apply operations on three sets (subset, overlapping and disjoint cases) using Venn diagrams.
- Solve problems on classification and cataloging by using Venn diagram for scenarios involving two sets and three sets.
- Verify and apply laws of union and intersection of three sets through analytical and Venn diagram method.
- Apply concepts from set theory to real world problems (such as in demographic classification, categorizing products in shopping malls and music playlist).
- Explain product, binary relations and identify domain and range of binary relations.
- Recognize that a relation can be represented by tables, ordered pairs and graphs.

ALLAH سبحانہ وتعالی has created this huge universe and a part of it is exposed to humans but a big part of it is still unknown to humans. It is estimated that there are 200 billions to 2 trillion galaxies in the observable universe. The adjoining figure shows a set of 10 naked eye galaxies.

SET

Definition:

A set is a well-defined collection of distinct objects.

The term well-defined, means that the objects follow a given discipline with which presence or absence of some object in the set is checked.

For instance, if we say that we have a collection of lighter stones, then this collection is not well defined. Instead of this, if we say that we have a collection of stones weighing less than 1kg, this collection is well defined.

A set may consist of objects of different types.

e.g. A set of luminous objects may contain a star, a moon, a tube light or a candle.

Similarly, a set of objects present in a library may include books, tables, chairs, newspapers, keys, locks, stock registers etc.

Example 1:

(a) A set of Prime numbers which are also even i.e. $\{2\}$

(b) A set of Pakistani currency i.e.

$\{5, 10, 20, 50, 100, 500, 1000, 5000\}$

(c) A set of flowers in my garden. i.e.

{Pansy, Lilly, Daisy, Jasmine, Tulip, Rose, Hibiscus}

(d) A set of Natural numbers less than 1 i.e. $\{\ \}$

(e) A set of all letters present in the word 'set' {s, e, t}

(f) A set of Natural numbers among $3, -9, 2, 3, 4, 2$ i.e. $\{2, 3, 4\}$.

The repeating elements are taken once only, since repetition is not allowed in a set.

> **Key Fact**
>
> A set is like a box with some stuff in it which is well defined. When you look inside the box, you should be able to tell if something's in it or not.

Mathematics as a Study of Patterns, Structures and Relationships

In mathematics, patterns are more than a beautiful design. Patterns follow a predictable rule and that rule allows us to predict what will come next.

For example:

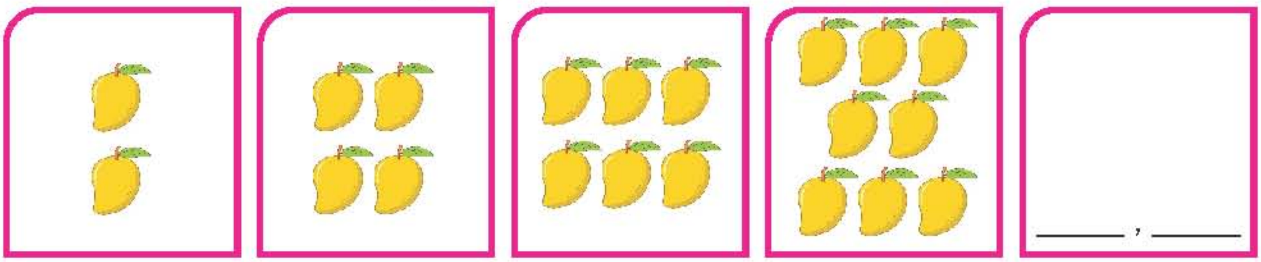
In the set of even numbers,

$$\{2, 4, 6, 8, \ldots\}$$

a pattern exists and one can determine the next number(s) in the set.

If we want to relate the above pattern with some real situation, we can ask the following question to the students:

What will be the number of mangoes in next two boxes?

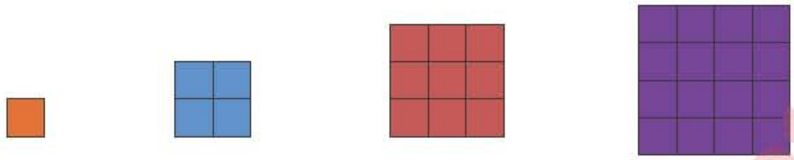


Students can easily predict the number of mangoes in the next two boxes.
Obviously, they will say:

Number of box	Number of mangoes
5	10
6	12

In the same way, we can relate the set of square of natural numbers with structural geometry as:

$$\{1^2, 2^2, 3^2, 4^2, \ldots\} = \{1, 4, 9, 16, \ldots\}$$



$1^2 = 1$ $2^2 = 4$ $3^2 = 9$ $4^2 = 16$

This example relates the patterns in numbers and geometry in the best way where the square numbers represent area of various geometrical shapes.

Key Fact

Mathematicians say that mathematics is the study of patterns in numbers and structure in geometry, and their relationships.

Check Point

Search number patterns in the set of first 100 natural numbers and relate patterns with some kind of geometrical shape or represent pattern in pictorial form.

Set Builder Form (Rule Method)

In set builder form, all the elements of a set are not listed, however we write the set by its defining rule. While writing a set in this method, some variable say x is chosen which represents all the elements of that set according to the defining rule.

e.g. A = Set of all integers, can be written in set builder form as

$A = \{x \mid x \in Z\}$ and read as

"A is the set of all elements x such that x belongs to Z"

Example 2: Write the following sets in the set builder form.

(i) B = Set of Prime numbers less than 17.

$B = \{x \mid x \in P \wedge x < 17\}$.

(ii) C = Set of multiples of 4 greater than or equal to 40.

$C = \{4x \mid x \in N \wedge x \geq 10\}$.

(iii) D = {1, 2, 3, 6}.
D = {x | x is a factor of 6}.

(iv) E = Set of squares of 1st three natural multiples of 10.
E = $\{(10x)^2 \mid x \in N \wedge 1 \le x \le 3\}$.

Example 3:

(i) The set $G = \{15x \mid x \in Z \wedge x \ge 1\}$, in descriptive form is written as
G = Set of integral multiples of 15, greater than or equal to 15.

(ii) The set $H = \{y \mid y \in W \wedge y^3 - 1 = 7\}$, in tabular form is H = {2}.

(iii) The set $I = \{x \mid x \in W \wedge -3 > x > -5\}$, in tabular form is I = { }.

(iv) The set $J = \left\{\frac{x}{2} \mid x \in E \wedge x > 0\right\}$, in tabular form is J = {1, 2, 3, 4, …}.

(v) The set $K = \{y \mid y \in Z^- \wedge y^2 = 9\}$, in descriptive method is
K = Set of negative integers containing '– 3' only.

Most Commonly used Sets of Numbers

i. N = Set of Natural numbers = {1, 2, 3, 4,……..}
ii. W = Set of Whole numbers = {0, 1, 2, 3, 4,…….}
iii. Z = Set of Integers = {0, ± 1, ± 2, ± 3,....}
iv. E = Set of Even numbers = {0, ± 2, ± 4, ± 6,….}
v. O = Set of Odd numbers = {± 1, ± 3, ± 5, ± ,…}
vi. P = Set of Prime numbers = {2, 3, 5, 7, 11, 13,…..}
vii. Q = Set of Rational numbers $= \left\{\frac{p}{q} / p, q \in Z \wedge q \ne 0\right\}$
viii. Q' = Set of Irrational numbers $= \left\{x / x \ne \frac{p}{q}, p, q \in Z \wedge q \ne 0\right\}$
ix. R = Set of Real numbers $= \{x / x \in Q \vee x \in Q'\}$

Check Point

Can we write set of Real numbers in tabular form? Justify!

The above mentioned sets from (i-vi) are first written in descriptive method and then in tabular form (Roster method).
However, the sets from (vii-ix) are first written in descriptive method then in set builder form.

Set Operations

Union of Sets

Let A and B be two given sets. Then union of A and B is the set of all those elements which are taken either from A or from B or from both.

The union of A and B is denoted by 'A∪B' and:

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$

e.g. If A = {1, 2, 3, 4, 5} and B = {3, 4, 1, 7, 6}, then:

$$A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$$

Intersection of Sets

Let A and B be given sets then the intersection of A and B is the set of elements which belong to both A and B.

The intersection of A and B is denoted by '$A \cap B$' and:

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

e.g. If $A = \{1, 2, 3, 4, 5\}$ and $B = \{3, 4, 1, 7, 6\}$, then:

$$A \cap B = \{1, 3, 4\}$$

Difference of Sets

Difference of two sets A and B, denoted by $A - B$ is a collection of those elements of A which are not present in B. For the two sets A and B:

$$A - B = \{x \mid x \in A \wedge x \notin B\}$$

Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{3, 4, 6, 7, 8\}$, then:

$A - B = \{1, 2, 3, 4, 5\} - \{3, 4, 6, 7, 8\} = \{1, 2, 5\}$

In general, $A - B \neq B - A$

Complement of a Set

Let $A \subset U$ (i.e. A is proper subset of the universal set). Then the set of all elements of U, which are not in A, is called complement of A.

Complement of A is denoted by A^c or $A' = U - A$ and is defined as:

$$\{x \mid x \in U \wedge x \notin A\}$$

e.g. If $U = \{1, 2, 3, 4, 5, 6, 7\}$, $A = \{2, 4, 6\}$, then:

$$A^c = U - A = \{1, 3, 5, 7\}$$

Key Fact

- The common elements of A and B are written once.
- Two sets A and B are said to be disjoint, if $A \cap B = \phi$
- The elements of A are never present in A^c and vice versa.
- Two sets are said to be overlapping, if neither set is subset of the other and their intersection is non-empty.

Union and Intersection of Three Sets

Disjoint Sets

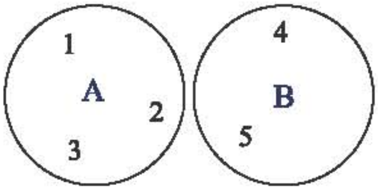
Two sets A and B are said to be disjoint if they have no common elements. i.e. $A \cap B = \phi$.

For example,

$A = \{1, 2, 3\}$ and $B = \{4, 5\}$ are disjoint sets as both sets have no common element. i.e.

$$A \cap B = \{1, 2, 3\} \cap \{4, 5\} = \phi$$

Venn diagram representing above disjoint sets is:



Overlapping Sets

Two sets A and B are called overlapping if:

(i) There is at least one element common in both the sets.

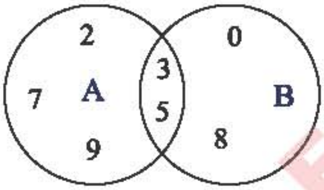
(ii) Neither of the sets is a subset of other set.

For example, the sets

$A = \{2, 3, 5, 7, 9\}$ and $B = \{0, 3, 5, 8\}$ are overlapping as

$A \cap B = \{3, 5\} \neq \phi$ and A and B are not subsets of each other.

Venn diagram representing above overlapping sets is:



Key Fact

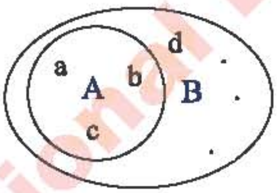
Venn diagrams are used to explain the whole set theory in a very simple way.

Subset

Set A is called subset of a set B if every element of set A is also an element of B.

For example, the set $A = \{a, b, c\}$ is a subset of $B = \{a, b, c, d, \dots\}$ as all elements of set A are also elements of set B.

Venn diagram representing above subset case is:



The pictorial representation of the relationship $A \subset U$ is called a Venn diagram. In a Venn diagram, universal set is represented by the interior of the rectangle, however inside the rectangle, the subsets are represented by the interior of any other closed shape like circles or ovals etc.

Union and Intersection of Three Sets using Venn Diagrams

To find the union and intersection of three sets say $A \cup B \cap C$, first we find $A \cup B$ or $B \cap C$ and then find the union or intersection with remaining set.

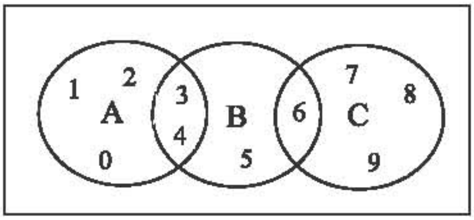
We use brackets to separate two sets from the third one because these represent different sets.

i.e. $(A \cup B) \cap C$ or $A \cup (B \cap C)$

Example 4:

Represent sets $A = \{0, 1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, $C = \{6, 7, 8, 9\}$ through Venn diagram.

Solution:

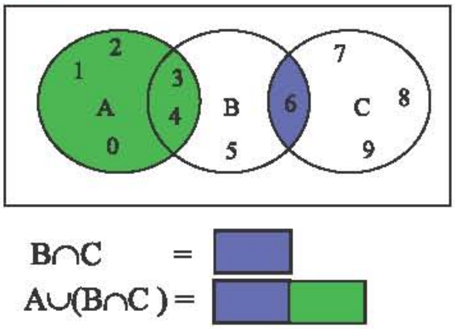


Example 5:

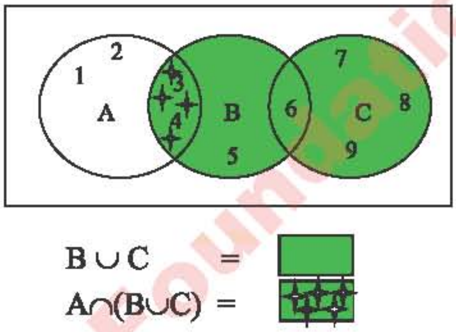
If A = {0, 1, 2, 3, 4}, B = {3, 4, 5, 6} and C = {6, 7, 8, 9}, find A ∪ (B ∩ C) and A ∩ (B ∪ C) through Venn diagram.

Solution:

(i)



(ii)



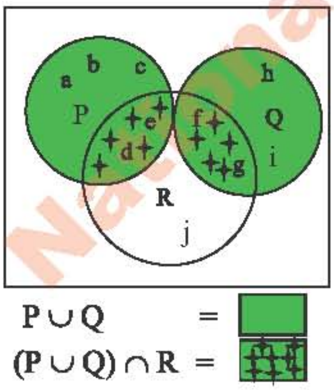
Example 6:

If P = {a, b, c, d, e}, Q = {f, g, h, i}, R = {d, e, f, g, j}

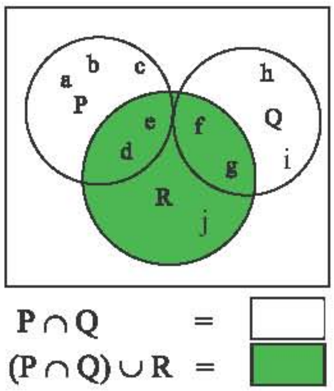
Find (P ∪ Q) ∩ R and (P ∩ Q) ∪ R through Venn diagram.

Solution:

(i)

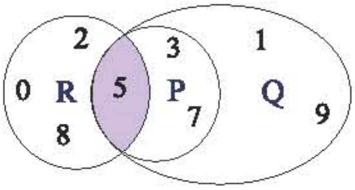


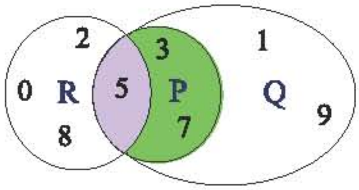
(ii)



Example 7: If P = {3, 5, 7}, Q = {1, 3, 5, 7, 9}, R = {0, 2, 5, 8}, then find P ∪ (Q ∩ R) using Venn diagram.

Solution: P = {3, 5, 7}, Q = {1, 3, 5, 7, 9}, R = {0, 2, 5, 8}





$Q \cap R = \{5\} =$

$P \cup (Q \cap R) = \{3, 5, 7\} =$

Verification of Associative Laws Using Venn Diagram

We illustrate the concept with the help of following examples.

Associative Property of Union

$$(A \cup B) \cup C = A \cup (B \cup C)$$

Proof:

Let $x \in (A \cup B) \cup C$

$\Rightarrow$ $x \in (A \cup B)$ or $x \in C$

$\Rightarrow$ $(x \in A$ or $x \in B)$ or $x \in C$

$\Rightarrow$ $x \in A$ or $(x \in B$ or $x \in C)$

$\Rightarrow$ $x \in A$ or $x \in (B \cup C)$

$\Rightarrow$ $x \in A \cup (B \cup C)$

$\Rightarrow$ $(A \cup B) \cup C \subseteq A \cup (B \cup C)$ (a)

Similarly, we can prove that:

$A \cup B \cup C) \subseteq (A \cup B) \cup C$ (b)

From (a) and (b), we have:

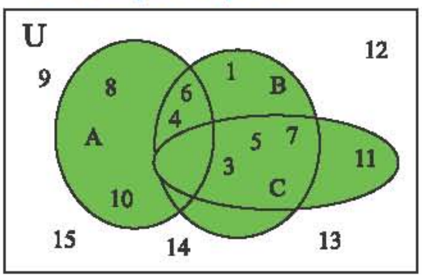
$$(A \cup B) \cup C = A \cup (B \cup C)$$

Example 8:

If U = {1, 2, 3, 4, 5,...,15}, A = {2, 4, 6, 8, 10}, B = {1, 2, 3, 4, 5, 6, 7} and C = {2, 3, 5, 7, 11}, then verify the associative property of union using Venn diagram.

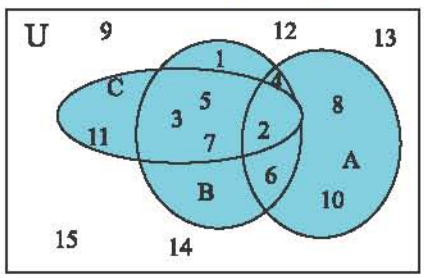
Solution:

LHS = $(A \cup B) \cup C$



$(A \cup B) \cup C =$

RHS = $A \cup (B \cup C)$



$A \cup (B \cup C) =$

From both the figures, it is observed that same region is shaded.

i.e. $(A\cup B)\cup C = A\cup (B\cup C)$

$\therefore$ Associative property of Union is verified.

Associative Property of Intersection

$$(A\cap B)\cap C = A\cap(B\cap C)$$

Proof:

Let $y \in (A\cap B)\cap C$

$\Rightarrow$ $y \in A\cap B$ and $y \in C$

$\Rightarrow$ $(y \in A$ and $y \in B)$ and $y \in C$

$\Rightarrow$ $y \in A$ and $(y \in B$ and $y \in C)$

$\Rightarrow$ $y \in A$ and $y \in (B\cap C)$

$\Rightarrow$ $y \in A\cap(B\cap C)$

$\Rightarrow$ $(A\cap B)\cap C \subseteq A\cap(B\cap C)$ (a)

Similarly, we can prove that:

$A\cap(B\cap C) \subseteq (A\cap B)\cap C$ (b)

From (a) and (b), we have:

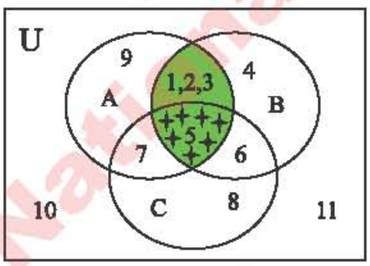
$$(A\cap B)\cap C = A\cap(B\cap C)$$

Example 9:

If $U = \{1, 2, 3, \ldots, 11\}$, $A = \{1, 2, 3, 5, 7, 9\}$, $B = \{1, 2, 3, 4, 5, 6\}$ and $C = \{5, 6, 7, 8\}$ then verify the associative property of intersection.

Solution:

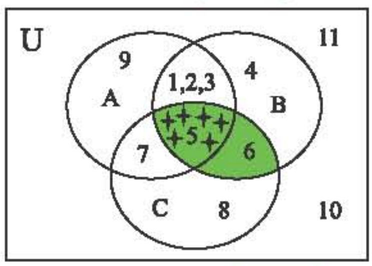
LHS = $(A\cap B)\cap C$



$A\cap B$ =

$(A\cap B)\cap C$ =

RHS = $A\cap(B\cap C)$



$B\cap C$ =

$A\cap(B\cap C)$ =

From both the figures it is observed that same regions are shaded.

i.e. $(A\cap B)\cap C = A\cap(B\cap C)$

$\therefore$ Associative property of intersection is verified.

Verification of Distributive Laws Using Venn Diagram

(a) Distributive Property of Union over Intersection

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proof:

Let $x \in A \cup (B \cap C)$

$\Rightarrow x \in A \text{ or } x \in (B \cap C)$

$\Rightarrow x \in A \text{ or } (x \in B \text{ and } x \in C)$

$\Rightarrow (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C)$

$\Rightarrow x \in (A \cup B) \text{ and } x \in (A \cup C)$

$\Rightarrow x \in (A \cup B) \cap (A \cup C)$

$\Rightarrow A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$ (a)

Again, let $y \in (A \cup B) \cap (A \cup C)$

$\Rightarrow y \in (A \cup B) \text{ and } y \in (A \cup C)$

$\Rightarrow (y \in A \text{ or } y \in B) \text{ and } (y \in A \text{ or } y \in C)$

$\Rightarrow y \in A \text{ or } (y \in B \text{ and } y \in C)$

$\Rightarrow y \in A \text{ or } y \in (B \cap C)$

$\Rightarrow y \in A \cup (B \cap C)$

$\Rightarrow (A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$ (b)

From (a) and (b), we have:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

(b) Distributive Property of Intersection over Union

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof:

Let $x \in A \cap (B \cup C)$

$\Rightarrow x \in A \text{ and } x \in (B \cup C)$

$\Rightarrow x \in A \text{ and } (x \in B \text{ or } x \in C)$

$\Rightarrow (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C)$

$\Rightarrow x \in (A \cap B) \text{ or } x \in (A \cap C)$

$\Rightarrow x \in (A \cap B) \cup (A \cap C)$

$\Rightarrow A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ (a)

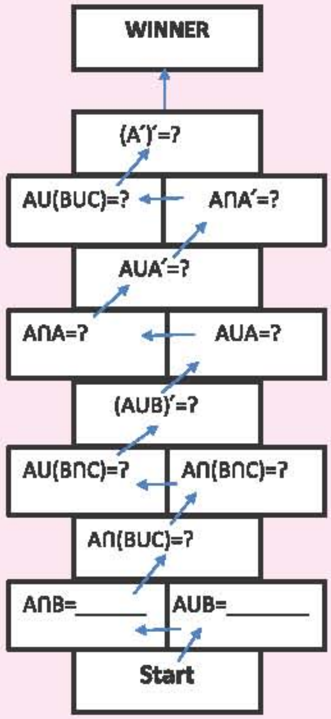
Again, let $y \in (A \cap B) \cup (A \cap C)$

$\Rightarrow y \in (A \cap B) \text{ or } y \in (A \cap C)$

$\Rightarrow (y \in A \text{ and } y \in B) \text{ or } (y \in A \text{ and } y \in C)$

Math Play Ground

1. Take students to the playground and make a hopscotch as shown:
2. Ask a student to start hoping and filling the blanks.



$\Rightarrow \quad y \in A$ and $(y \in B$ or $y \in C)$

$\Rightarrow \quad y \in A$ and $y \in (B \cup C)$

$\Rightarrow \quad y \in A \cap (B \cup C)$

$\Rightarrow \quad (A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C) \qquad$ (b)

From (a) and (b), we have:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Example 10: Verify through Venn diagram

(a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

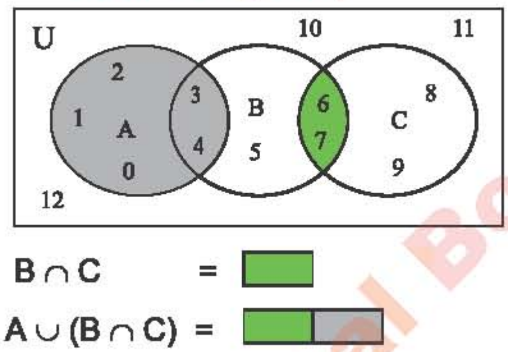
When $U = \{x : x \in W \wedge x \leq 12\}$, $A = \{x : x \in W \wedge x \leq 4\}$

$B = \{y : y \in N \wedge 3 \leq y \leq 7\}$ and $C = \{z : z \in N \wedge 3 \leq z \leq 7\}$

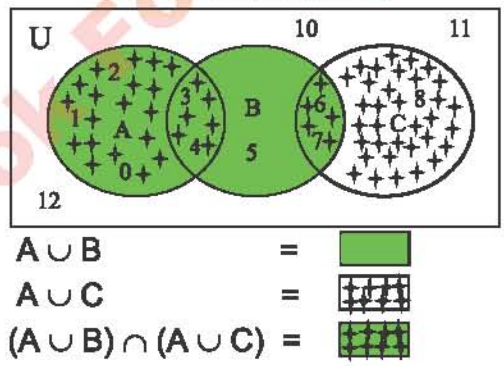
Solution:

(a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

LHS = A ∪ (B ∩ C)

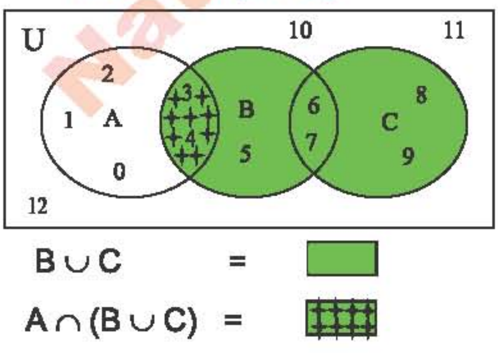


RHS = (A∪B) ∩ (A∪C)

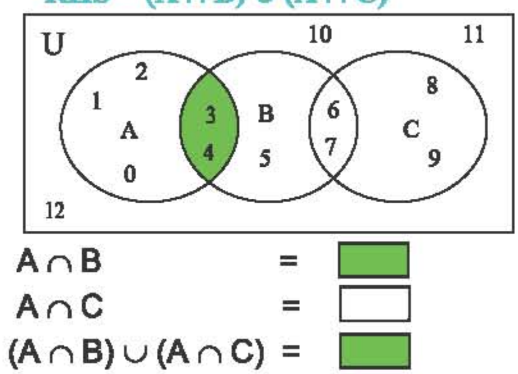


(b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C$

LHS = A ∩ (B ∪ C)



RHS = (A ∩ B) ∪ (A ∩ C)



Key Fact

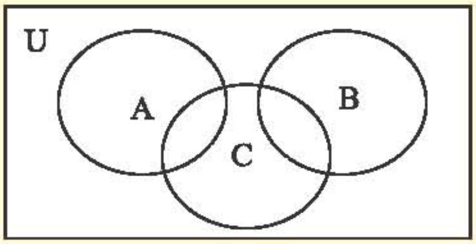
(a) If A = {0, 1, 3, 5, 7}, $B = \{x \mid x \in N \wedge x \leq 5\}$ and C = {1, 2, 3, 4, 6, 12}, then find $(A \cup B) \cap C$ using Venn diagrams.

(b) If A = {0, 2, 4, 6}, B = {1, 3, 5, 7} and C = {1, 2, 3, 6}, then find $(A \cap B) \cap C$, $A \cap (B \cap C)$ and $(A \cup B) \cap C$ using Venn diagram.

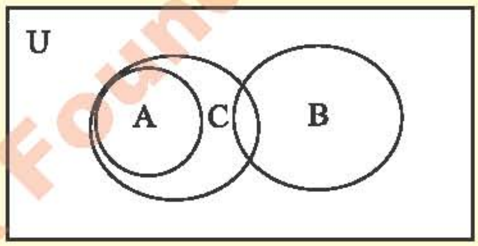
EXERCISE 3.1

1. Shade $A \cup (B \cap C)$, $A \cap (B \cup C)$, $(A \cup B) \cup C$ and $A \cap (B \cap C)$ using following Venn diagrams.

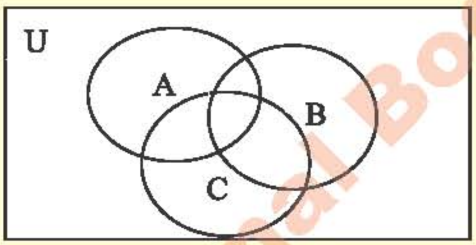
(i)



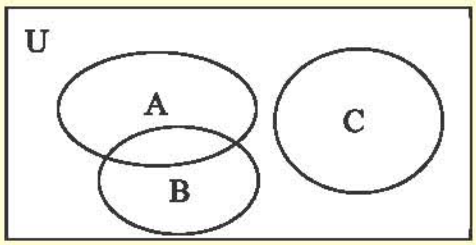
(ii)



(iii)



(iv)



2. If X = {a, b, c, d, e}, Y = {a, c, e}, Z = {g, h, i, j} then, find the following using Venn diagram.

(i) $(X \cup Y) \cup Z$ (ii) $X \cup (Y \cup Z)$ (iii) $(X \cap Y) \cap Z$
(iv) $X \cap (Y \cap Z)$ (v) $(X \cup Y) \cap Z$ (vi) $(X \cap Y) \cup Z$

3. Verify associative law of union and intersection by using diagrams of question 1.

4. Verify:
   (i) distributive property of union over intersection,
   (ii) distributive property of intersection over union,
   by using diagrams of question 1.

5. Prove by using Venn diagram:
   (a) $(P \cup Q) \cup R = P \cup (Q \cup R)$ (b) $(P \cap Q) \cap R = P \cap (Q \cap R)$
   when (i) P = {0,1 ,2, 3}, Q = {2, 3, 4, 5, 6}, R = {5, 6, 7, 8, 9}

(ii) P = {m, n, o, p, q}, Q = {r, s, t, u}, R = {t, u, v, w}

6. Verify $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$ using Venn diagram for the following sets.
$X = \{-1, -2, -3\}$, $Y = \{0, 1, 2, 3\}$, $Z = \{0, \pm 1, \pm 2, \pm 3\}$

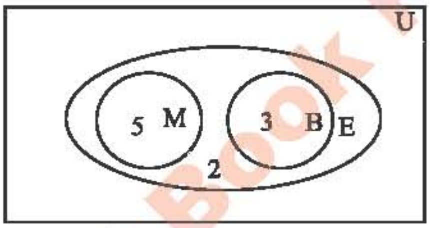
7. Verify $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$
X = Set of first three Vowels, Y = Set of letters of the word " energy",
Z = Set of letters of the word "algebra"

Application of Venn Diagrams

Venn diagram is a practical mathematical tool for solving real world problems of set theory. Few of the examples depict the vital role of the Venn diagrams in problem solving.

Example 11: Among the ten teachers of a secondary school, five teach Mathematics and three teach Biology. However, all these teachers also teach English? Show the data by Venn diagram. Also find how many teachers teach only English.

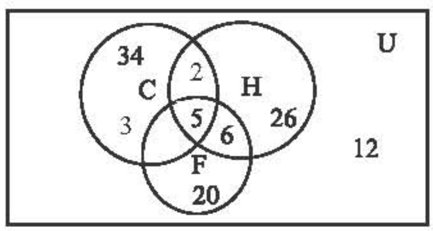
Solution: If E represents set of English Teachers, M represents set of Mathematics Teachers, B represents the set of Biology Teachers and then both M and B are the subsets of E, as shown in the figure below.



From the Venn diagram it is observed that two of the teachers only teach English.

Example 12: In a survey, people were asked whether they like cricket, hockey or football. Using the Venn diagram find the number of people playing:

(i) only cricket,
(ii) hockey and football,
(iii) all three games,
(iv) either of the three games,
(v) neither of the three games,
Also find number of people surveyed.



Solution: From Venn diagram, it is clear that:

(i) Number of people who play only cricket = 34
Which shows $n(C - C \cap H \cap F)$

(ii) Number of people who play both hockey and football = 6 + 5 = 11
Which shows $n(H \cap F)$

(iii) Number of people who play all three games = 5
Which shows $n(C \cap H \cap F)$

(iv) Number of people who play either of the games = 34 + 5 + 6 + 26 + 20 = 91
Which shows $n(C \cup H \cup F)$

(v) Number of people who do not play any game = 12
Which shows $n(U - C \cup H \cup F)$

Total number of people = n(U) = 34 + 5 + 6 + 26 + 20 + 12 = 103

Application of Set Theory

Applications of set theory are most frequently used in science and mathematics fields like biology, chemistry, and physics as well as in computer and electrical engineering. These applications range from forming logical foundations for all branches of mathematics. Therefore, understanding set theory is crucial for learning many subjects.

Following formulae are helpful in the set theory.

(i) For any two overlapping sets A and B:

- $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
- $n(A - B) = n(A \cup B) - n(B)$
- $n(A - B) = n(A) - n(A \cap B)$
- $n(A \cup B) = n(A - B) + n(A \cap B) + n(B - A)$

Key Fact

For any sets A and B:

$A \cup A = A$, $A \cap A = A$, $A \cap \emptyset = \emptyset$

$A \cup \emptyset = A$, $A \cap B \subseteq A$, $A \subseteq A \cup B$

(ii) For any two sets A and B that are disjoint:

- $n(A \cup B) = n(A) + n(B)$
- $n(A - B) = n(A)$

(iii) For any three sets A, B and C:

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$

Example 13:

In a class of 80 students, 40 like English, 34 like Mathematics and 9 like both. How many students like either of both subjects and how many like neither?

Solution:

Total number of students = n(T) = 80

Number of students that like English = n(E) = 40

Number of students that like Mathematics = n(M) = 34

Therefore, total number of students that like both subjects is:

$$n(E \cup M) = n(E) + n(M) - n(E \cap M)$$
$$= 40 + 34 - 9 = 65$$

Number of students that do not like both subjects is:

$$n(T) - n(E \cup M) = 80 - 65 = 15$$

EXERCISE 3.2

1 Let A and B be two finite sets such that $n(A) = 24$, $n(B) = 18$ and $n(A \cup B) = 31$. Find $n(A \cap B)$.

2. If $n(A - B) = 23$, $n(A \cup B) = 44$ and $n(A \cap B) = 2$, then find $n(B - A)$. Also find $n(B)$. (Hint: $n(B) = n(A \cap B) + n(B - A)$)

3. In a group of 30 Mathematics students, 20 like Algebra and 15 like both Geometry and Algebra. Show the data by Venn diagram. Also find how many students like Geometry.

4. In a street with 50 houses, 25 houses have lawns, 32 houses have car porch and 15 houses have both lawn and car porch. Show the data by Venn diagram. Also find how many houses have neither lawn nor porch.

5. In a survey of 940 children, 400 students were found studying at primary level, 240 students at elementary and 175 at secondary level. Create a Venn diagram to illustrate this information. How many children were found out of school?

6. ABC Dairy polls its customers on their favorite flavor: chocolate, vanilla or mango? 100 customers said they like mango flavor, 90 customers said they like vanilla, 40 polled for chocolate, 20 customers liked both mango and vanilla while 14 liked both chocolate and vanilla. How many customers said they like:
(i) only mango? (ii) only vanilla (iii) only chocolate

7. In a survey of university 200 students were interviewed. It was found that:
42 students have laptops, 80 students have cell phones, 100 students have iPods,
23 students have both a laptop and a cell phone, 10 students have both a laptop and iPod,
14 students have both a cell phone and iPod and 8 students have all three items.
(a) How many students have only cell phone?
(b) How many students have none of the three items?
(c) How many students have both iPod and laptop but not cellphone?

8. In a girl college, every student plays either badminton or table tennis or both. If 350 students play badminton, 280 play table tennis and 150 play both. Find how many students are there in the college?

9. Among 50 students, 8 are learning both English and Chinese languages. A total of 26 students are learning English. If every student is learning at least one language, how many students are learning Chinese?

10. Out of 70 people, 48 like tea and 40 like coffee and each person likes at least one of the two drinks. How many like both tea and coffee?

11. There are 46 students in science group and 50 students in arts group. Find the number of students who are either in science or arts group.

12. In a group of people, 52 people can speak Arabic and 112 can speak French. How many can speak Arabic only? How many can speak French only if 12 of them can speak both languages? How many people were in the group?
13. In a high school, 360 students like reading story books, 170 like practical activities and 150 like both. Find
   (i) The number of students who like reading story books only.
   (ii) The number of students who like only practical activities.
   (iii) The total number of students in the school.
14. In a survey of 60 people, it was found that 25 people watch channel A, 16 watch channel B, 13 watch channel C, 4 watch both A and B, 7 watch both B and C, 8 watch both A and C, 3 watch all three channels. Find the number of people who watch at least one of the channels? Also find number of people who do not watch these channels.

Binary Relations

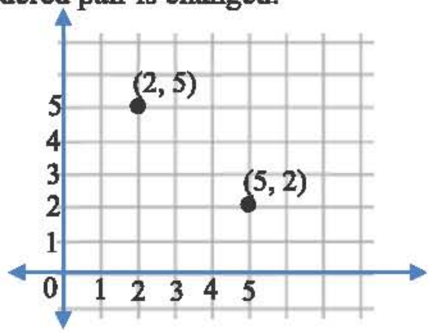
Ordered Pair

Pairs of two numbers in which order of numbers is not invertible, is called an ordered pair. The numbers in an ordered pair are written within small brackets (parenthesis) and are separated by comma.

For example, (a, b) is an ordered pair in which a is called first element and b is called second element. By interchanging the positions of elements, the ordered pair is changed.

As in geometry, position of a point is determined by ordered pair, therefore (2, 5) and (5, 2) represent two different points.

Thus, $(2, 5) \neq (5, 2)$



Equality of Two Ordered Pairs

Two ordered pairs (a, b) and (c, d) are equal if:

$$a = b \text{ and } c = d$$

Key Fact
- $(a, b) \neq (b, a)$
- $(a, b) = (c, d) \Leftrightarrow a = c \;\&\; b = d$

Example 15:

Find the values of x and y when $(x - 3y, 5x + 1) = (4, 6)$

Solution: By the equality of ordered pairs, we have:

$$x - 3y = 4 \qquad \text{(i)}$$
$$5x + 1 = 6 \qquad \text{(ii)}$$

From equation (ii), $5x = 6 - 1$

$$5x = 5 \quad \Rightarrow \quad x = 1$$

Substituting $x = 1$ in equation (i), we get:

$$1 - 3y = 4 \Rightarrow -3y = 3$$
$$\Rightarrow y = -1$$

Check Point

Find a and b when:
$(a + 1, 4) = (2, b - 3)$

Cartesian Product of Sets

If A and B are two non-empty sets, then Cartesian product $A \times B$ is the set of ordered pairs (x, y) such that $x \in A$ and $y \in B$. Mathematically:

$A \times B = \{(x, y) \mid x \in A \wedge y \in B\}$

Key Fact

- $n(A \times B) = n(A) \times n(B)$
- $A \times B = \phi$ if either $A = \phi$ or $B = \phi$

Similarly, $B \times A = \{(y, x) \mid y \in B \wedge x \in A\}$

e.g. If $A = \{0, 1, 2\}$, $B = \{3, 4\}$

then $A \times B = \{0, 1, 2\} \times \{3, 4\}$

$= \{(0, 3), (0, 4), (1, 3), (1, 4), (2, 3), (2, 4)\}$

$A \times B$ can also be represented through table as follows:

A × B	0	1	2
3	(0, 3)	(1, 3)	(2, 3)
4	(0, 4)	(1, 4)	(2, 4)

In the same way, we can find $B \times A$ as:

$B \times A = \{3, 4\} \times \{0, 1, 2\}$

$= \{(3, 0), (3, 1), (3, 2), (4, 0), (4, 1), (4, 2)\}$

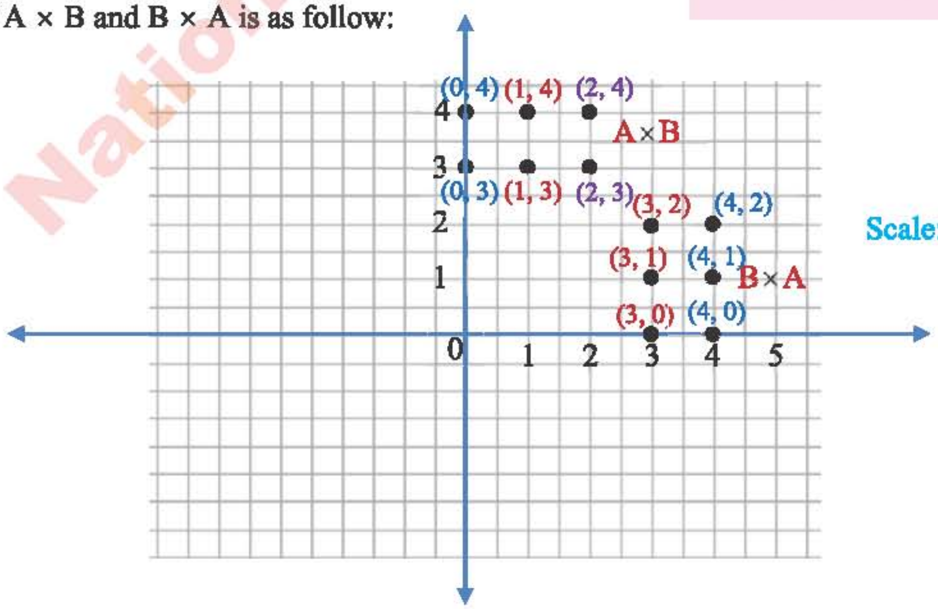
Table for $B \times A$ is:

B × A	3	4
0	(3, 0)	(4, 0)
1	(3, 1)	(4, 1)
2	(3, 2)	(4, 2)

Key Fact

- Each element of the set $A \times B$ is called an ordered pair.
- The ordered pair $(a, 1)$ cannot be written as $(1, a)$.
- The number of subsets of $A \times B = 2^{n(A \times B)}$.

Graph of $A \times B$ and $B \times A$ is as follow:



Scale: 2 squares = 1

From the definition, table and graph, we see that:

$$A \times B \neq B \times A$$

Key Fact

- In general $A \times B \neq B \times A$
- $A \times B = B \times A$ if and only if $A = B$.
- $n(A \times B) = n(B \times A)$

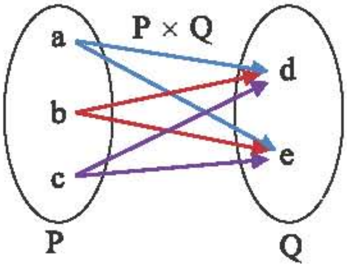
There are many subsets of $A \times B$
e.g. $R_1 = \phi$, $R_2 = \{(0, 3)\}$, $R_3 = \{(1, 3), (0, 4)\}$, …

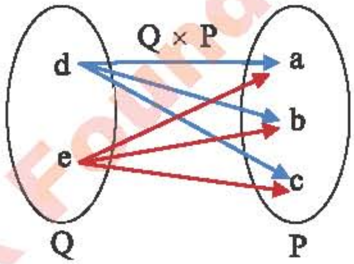
Example 16:

Exhibit $P \times Q$ and $Q \times P$ by arrow diagram when $P = \{a, b, c\}$ and $Q = \{d, e\}$.

Solution:

$P \times Q = \{(a, d), (a, e), (b, d), (b, e), (c, d), (c, e)\}$
$Q \times P = \{(d, a), (d, b), (d, c), (e, a), (e, b), (e, c)\}$





EXERCISE 3.3

1. Find the values of unknowns when:
   (i) $(a, -b) = (7, 1)$ (ii) $(2a, 2b + 3) = (-10, -b)$ (iii) $(2a - 4, 6) = (8, -b + 1)$
   (iv) $(x + 2y, y - 3) = (2, 5)$ (v) $(2x - y, y - 3x) = (4, 2)$
   (vi) $(4x + 6y, x - 12y) = (6, -3)$ (vii) $(5x + y, -x + y) = (6, 1)$
2. Let $A = \{1, 4, 8\}$ and $B = \{1, 0\}$. Find:
   (i) $A \times B$ (ii) $B \times A$ (iii) $A \times A$ (iv) $B \times B$
   How many elements are there in $A \times B$, $B \times A$, $A \times A$ and $B \times B$?
3. Let $E = \{1, 3\}$ and $F = \{4, 6, 8\}$. Express $E \times F$, $F \times E$, $E \times E$, $F \times F$ graphically.
4. If $L \times M = \{(0, 2), (0, 3), (0, 4), (1, 2), (1, 3), (1, 4)\}$, then find sets L, M and $M \times L$.
5. Given that $A = \{1, 3, 5\}$, $B = \{2, 4\}$, $C = \{6, 7\}$.
   (i) Find $A \times (B \cup C)$ (ii) Find $(A \times B) \cup (A \times C)$
   (iii) Verify $A \times (B \cup C) = (A \times B) \cup A \times C)$
6. Given that $D = \{a, e, i\}$, $E = \{a, c\}$, $F = \{b, c\}$.
   (i) Find $D \times (E \cap F)$ (ii) Find $(D \times E) \cap (D \times F)$
   (iii) Verify $D \times (E \cap F) = (D \times E) \cap (D \times F)$

7. Given that $A = \{x / x \in N, x < 3\}$, $B = \{y / y \in W, y < 2\}$, $C = \{0, 2, 4\}$.
   (i) Verify $A \times (B - C) = (A \times B) - (A \times C)$
   (ii) Verify $(A - B) \times C = (A \times C) - (B \times C)$
8. Let $X = \{x / x \in W, x \leq 2\}$ and $Y = \{-1, -2, -3\}$. Exhibit $X \times Y$ and $Y \times X$ by arrow diagram.

Binary Relation

A binary relation R in the set A × B is a subset of the Cartesian product A × B.
Symbolically R is the relation in a set A × B if and only if $R \subseteq A \times B$.
If R is elation from A to B, then:

$$R = \{(a, b) / a \in A, b \in B\}$$

A binary relation can also be taken from only one set after taking Cartesian product of the set with itself e.g. from A × A
If R_1 is elation from A to A, then:

$$R_1 = \{(a, b) / a \in A, b \in A\}$$

Key Fact
- A binary relation is a set of ordered pairs.
- Number of binary relations in $A \times B = 2^{n(A \times B)}$

Example 17:
If A = {5, 10, 15, 20, 25} then find the number of binary relations in A.

Solution: Number of elements in $A = n(A) = 5$
Number of elements in $A \times A = n(A \times A) = 5 \times 5 = 25$
Number of binary relations in A (or A × A) $= 2^{25}$

Example 18:

If $P = \{2, 3\}$, $Q = \left\{\frac{1}{2}, \frac{1}{3}\right\}$, then find all possible binary relations in $P \times Q$.

Solution:
Number of elements in $P = n(P) = 2$
Number of elements in $Q = n(Q) = 2$
Number of elements in $P \times Q = n(P \times Q) = n(P) \times n(Q)$

$$= 2 \times 2 = 4$$

Number of binary relations in $P \times Q = 2^{n(P \times Q)}$

$$= 2^{2 \times 2} = 2^4 = 16$$

Now $P \times Q = \left\{\left(2, \frac{1}{2}\right), \left(2, \frac{1}{3}\right), \left(3, \frac{1}{2}\right), \left(3, \frac{1}{3}\right)\right\}$. Here,

$$R_1 = \phi,\ R_2 = \left\{\left(2, \frac{1}{2}\right)\right\},\ R_3 = \left\{\left(2, \frac{1}{3}\right)\right\},\ R_4 = \left\{\left(3, \frac{1}{2}\right)\right\},\ R_5 = \left\{\left(3, \frac{1}{3}\right)\right\},$$

$$R_6 = \left\{\left(2, \frac{1}{2}\right), \left(2, \frac{1}{3}\right)\right\},\ R_7 = \left\{\left(2, \frac{1}{2}\right), \left(3, \frac{1}{2}\right)\right\},\ R_8 = \left\{\left(2, \frac{1}{2}\right), \left(3, \frac{1}{3}\right)\right\},$$

$$R_9=\left\{\left(2,\frac{1}{3}\right),\left(3,\frac{1}{2}\right)\right\},\ R_{10}=\left\{\left(2,\frac{1}{3}\right),\left(3,\frac{1}{3}\right)\right\},\ R_{11}=\left\{\left(3,\frac{1}{2}\right),\left(3,\frac{1}{3}\right)\right\},$$

$$R_{12}=\left\{\left(2,\frac{1}{2}\right),\left(2,\frac{1}{3}\right),\left(3,\frac{1}{2}\right)\right\},\ R_{13}=\left\{\left(2,\frac{1}{2}\right),\left(2,\frac{1}{3}\right),\left(3,\frac{1}{3}\right)\right\}$$

$$R_{14}=\left\{\left(2,\frac{1}{2}\right),\left(3,\frac{1}{2}\right),\left(3,\frac{1}{3}\right)\right\},\ R_{15}=\left\{\left(2,\frac{1}{3}\right),\left(3,\frac{1}{2}\right),\left(3,\frac{1}{3}\right)\right\}$$

$$R_{16}=\left\{\left(2,\frac{1}{2}\right),\left(2,\frac{1}{3}\right),\left(3,\frac{1}{2}\right),\left(3,\frac{1}{3}\right)\right\}$$

Example 19:

Let A = {1, 2, 3}, B = {0, 1, 3} and R = {(a, b) / $a \in$ A, $b \in$ B} and $a > b$}, then find R and show it by arrow diagram.

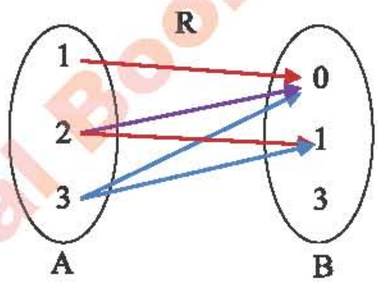
Solution:

A = {1, 2, 3}, B = {0, 1, 3}

A × B = {(1, 0), (1, 1), (1, 3), (2, 0), (2, 1), (2, 3), (3, 0), (3, 1), (3, 3)}

Now R = {(a, b) / $a \in$ A, $b \in$ B} and $a > b$}

R = {(1, 0), (2, 0), (2, 1), (3, 0), (3, 1)}



Domain of Binary Relation

Set of all the first elements of ordered pairs in a binary relation is called domain of that binary relation.

In example 18, Domain of R_{11} = {3} and Domain of R_{14} = {2, 3}.

Range of Binary Relation

Set of all the second elements of ordered pairs in a binary relation is called range of that binary relation.

In example 18, Range of $R_6=\left\{\frac{1}{2},\frac{1}{3}\right\}$ and Range of $R_7=\left\{\frac{1}{2}\right\}$.

Example 20:

(a) If A = Set of Natural numbers and $R = \{(x, y) \mid x \in A \wedge y \in A\}$ i.e. $R \subseteq A \times A$

Then find the domain and range of R.

(b) If $T = \{0, \pm 1, \pm 2\}$ and $R_1 = \{(x, y) \mid x \in T \wedge y \in T \wedge x + y = 0\}$, then find the Dom R and Range R.

(c) If $E = \{2, 4, 6\}$, $F = \{0, 1, 2\}$ and $R_2 = \{(x, y) \mid x \in E, y \in F \wedge x + y = 6\}$, then

(i) Write $E \times F$ **(ii)** Write R_2 in tabular form **(iii)** Find Dom R_2 and Range R_2.

Solution:

(a) $R = \{(1, 1), (1, 2), (1, 3), \ldots, (2, 1), (2, 2), (2, 3), \ldots (3, 1), (3, 2), (3, 3), \ldots\}$

So, Domain of $R = \{1, 2, 3, \ldots\}$ = Set of Natural numbers

Range of $R = \{1, 2, 3, \ldots\}$ = Set of Natural numbers

$\therefore$ Dom R = Range R = A

(b) $R_1 = \{(0, 0), (-1, 1), (1, -1), (-2, 2), (2, -2)\}$

Dom $R_1 = \{0, \pm 1, \pm 2\}$, Range $R_1 = \{0, \pm 1, \pm 2\}$

(c) $E \times F = \{(2, 0), (2, 1), (2, 2), (4, 0), (4, 1), (4, 2), (6, 0), (6, 1), (6, 2)\}$

$R_2 = \{(4, 2), (6, 0)\}$

Dom $R_2 = \{4, 6\}$, Range $R_2 = \{0, 2\}$

Key Fact

$R_1 = \phi$ is called a void relation.

Inverse Relation

Let $R = \{(a, b) / a \in A, b \in B\}$ be a relation from A to B then the inverse of R is defined by:

$R^{-1} = \{(b, a) / b \in B, a \in A\}$

Example 21:

Given that $A = \{0, 2, 3\}$, $B = \{0, 2, 4, 6, 9, 16\}$ and $R = \{(x, y) \mid x \in A \wedge y \in B \wedge x^2 = y\}$.

Verify: Dom R^{-1} = Range R and Range R^{-1} = Dom R

Solution:

Given $A = \{0, 2, 3\}$, $B = \{0, 2, 4, 6, 9, 16\}$ and $R = \{(x, y) / x \in A, y \in B\} \wedge x^2 = y\}$

R in tabular form is:

$R = \{(0, 0), (2, 4), (3, 9)\}$

Dom $R = \{0, 2, 3\}$ and Range $R = \{0, 4, 9\}$

Inverse of R is:

$R^{-1} = \{(0, 0), (4, 2), (9, 3)\}$

Dom $R^{-1} = \{0, 4, 9\}$ and Range $R^{-1} = \{0, 2, 3\}$

Which shows that:

Dom R^{-1} $\{0, 4, 9\}$ = Range R and Range $R^{-1} = \{0, 2, 3\}$ = Dom R

Key Fact

- Dom R^{-1} = Range R
- Range R^{-1} = Dom R

EXERCISE 3.4

1. Find the number of binary relations in the following cases.
   (i) $A = \{1, 3\}, B = \{0, 2, 4\}$ (ii) $n(C) = 7$ (iii) $D = \{1, 3, 5\}$
2. Find all possible binary relations in the following cases mentioning the number of binary relations in each case.
   (i) $A = \left\{\sqrt{2}, \sqrt{3}, \sqrt{5}\right\}$, $B = \left\{\sqrt[3]{5}\right\}$ (ii) $C = \{\pi, e\}$ (iii) $D = \{5\}, E = \{1, 10\}$
3. If $P = \{7, 8, 9\}$ then find 2 binary relations from P to P. Also find domain and range of each relation.
4. Let $H = \{5, 6, 7, 8, 9\}$ and $G = \{5, 7, 9, 11\}$. Write the following relations from H to G in tabular form.
   (i) 'is equal to' (ii) 'is less than' (iii) 'is greater than'
   (iv) 'is one less than' (v) 'is one greater than' (vi) 'is two less than'
5. Let $C = \{2, 4, 6\}, D = \{4, 6, 8, 9, 12\}$ and $R = \{(x, y) / x \in C, y \in D\} \wedge x$ is factor of $y\}$.
   (i) Write R in tabular form. (ii) Find domain and range of R.
   (iii) Find R^{-1}. (iv) Represent R by arrow diagram.
6. Let $R = \{(2, 0), (4, 2), (6, 4), (8, 6), (10, 8)\}$
   (i) Write R in set builder form. (ii) Find domain and range of R.
   (iii) Write R^{-1} in tabular and set builder from.
   (iv) Represent R and R^{-1} by arrow diagram.
7. Let $A = \{0, 1, 3\}$ and $B = \{1, 2, 3, 5, 7\}$. Write $R = \{(x, y) \mid x \in A, y \in B \wedge y = 2x + 1\}$ in tabular form. Also find R^{-1}.
8. If $S = \{1, 2, 4, 8\}, T = \{3^0, 3^1, 3^2\}$, then write the following binary relations in tabular form.
9. Find the domain and range in each case.
   (i) $R_1 = \{(x, y) \mid x \in S, y \in T \wedge x = y\}$ (ii) $R_2 = \{(x, y) \mid x \in S, y \in T \wedge y < x\}$
   (iii) $R_3 = \{(x, y) \mid x \in S, y \in T \wedge x + y \in E\}$ (iv) $R_4 = \{(x, y) \mid x \in S, y \in T \wedge x \times y \in O\}$
   (v) $R_5 = \{(x, y) \mid x \in S, y \in T \wedge y > 2x\}$

KEY POINTS

- If A and B are any two sets, then the set consisting of all the elements of these two sets is called union of these two sets.
- If A and B are any two sets then the set consisting of all the common elements of these two sets is called intersection of these two sets.
- The pictorial representation of any set is called a Venn diagram.
- We can prove associative laws and distributive laws by using Venn diagrams.
- Any subset of the Cartesian product $A \times B$ is a binary relation.

MISCELLANEOUS EXERCISE 3

1. Encircle the correct option.

i. Set builder form of $A - B$ is:

(a) $\{x \mid x \in A\}$ (b) $\{x \mid x \in A \wedge x \notin B\}$ (c) $\{x \mid x \in A \wedge x \in B\}$ (d) $\{x \mid x \in B\}$

ii. If $A \cup B = A$ and $A \cap B = B$, then:

(a) $A \subset B$ (b) $A \not\supseteq B$ (c) $A \supseteq B$ (d) $A \neq B$

iii. If $A \subset B$ then $A - B =$

(a) A (b) B (c) ϕ (d) $B - A$

iv. If $A - B = B - A = \phi$, then:

(a) $A = B$ (b) $B \subseteq A$ (c) $A \subseteq B$ (d) all a, b & c

v. Set of common elements of A and A^c is _____ set.

(a) infinite (b) null (c) universal (d) singleton

vi. Set of real numbers can be written in:

(a) tabular form. (b) descriptive form.
(c) set builder form. (d) both b and c.

vii. If $R = \{(2, 1), (4, 3), (2, 2)\}$, then Dom R =

(a) $\{2, 4, 2\}$ (b) $\{2, 4\}$ (c) $\{1, 3, 2\}$ (d) none of these

viii. If $R = \{(a, b) / a, b \in Z \wedge a + b = 0\}$, then:

(a) Dom R $\subset$ Range R (b) Range R $\subset$ Dom R
(c) Dom R = Range R (d) none of these.

ix. If $R = \{(a, b) / a, b \in N \wedge a \times b = 12\}$ then tabular form of R is:

(a) $\{(1, 12), (2, 6), (3, 4), (4, 3), (6, 2), (12, 1)\}$ (b) $\{(1, 12), (2, 6), (3, 4)\}$
(c) $\{(12, 1), (6, 2), (4, 3)\}$ (d) $\{(1, 12), (4, 3), (2, 6)\}$

x. If $n(A) = p$, then $n(A \times A) =$

(a) p (b) 2p (c) $2p^2$ (d) p^2

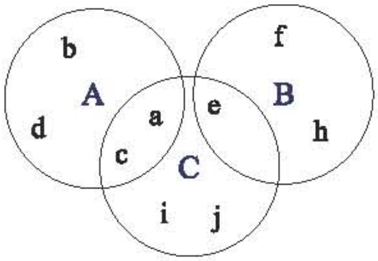
xi. If $n(B) = t$, then number of binary relations in $B \times B$ is:

(a) t (b) t^2 (c) 2^t (d) 2^{t^2}

2. If $R = \{a, f, h, s\}$ and $S = \{b, e, j, n\}$, then

(i) Find the number of binary relations in $R \times S$.
(ii) Write any 3 binary relations from $R \times S$.
(iii) Write a binary relation whose domain is equal to set R.
(iv) Write a binary relation whose range is equal to set S.
(v) Write a binary relation whose domain is equal to set R and range is equal to set S.

3. Shade $A \cup (B \cap C)$, $A \cap (B \cup C)$, $A - (B \cap C)$ and $B - (A \cup C)$, in the following Venn diagram.



4. Verify associative properties of union and intersection through Venn diagram for
$X = \{2x \mid x \in N \wedge x < 20\}$, Y = Set of first 6 natural multiples of 3, $Z = \{6x \mid x \in W \wedge x < 20\}$

5. Verify distributive law of union over intersection through Venn diagram for the following sets.

$A = \{1, 2, 3, ...\}$, $B = \{-1, -2, -3, ...\}$, $C = \{0, \pm 1, \pm 2, \pm 3, ...\}$

6. Verify distributive law of intersection over union through Venn diagram for the following sets.

$P = \{a, b, c, d, e\}$, $Q = \{c, d, e, f\}$, $R = \{g, h, i, j, k\}$

7. Create a Venn diagram to illustrate the following information regrading the subsets A and B in the universal set.
   (i) $n(A) = 60, n(B) = 48, n(A \cap B) = 20, n(U) = 90$
   (ii) $n(A) = 34, n(B) = 52, n(A \cup B) = 60, n(U) = 85$

8. 10 boys participated in a Qiraat competition. Among them, Haani, Zubair and Haider recited in Naafi Qiraat style, Abdullah, Umer, Bilal and Ali recited in Al-Kissai Qiraat style while Hassan, Jaffer and Usman recited in Al-Kufi Qiraat style. Represent boys participated in Naafi, Al-Kissai and Al-Kufi styles by sets A, B and C respectively. Find:
   (i) Find tabular form of A, B and C.
   (ii) Draw Venn diagram of situation.
   (iii) Find $A \cap (B \cap C)^c$, $(A \cup B) \cap C^c$, $A - (B \cap C)$ and $A - (A \cup B)$.

9. 100 candidates were appeared in an examination. Out of which 45 candidates passed in Mathematics, 40 in Science and 50 in Health. If 12 were passed in Mathematics and Science, 15 in Science and Health, 20 in Health and Mathematics and 5 were passed in all three subjects.
   (i) Illustrate the above information by drawing a Venn diagram.
   (ii) How many candidates were passed at least one subject?
   (iii) How many candidates did not pass any subject?

UNIT

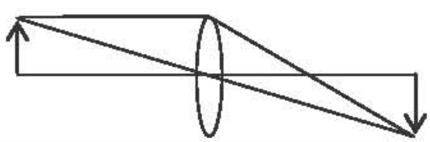
04

FACTORIZATION AND ALGEBRAIC MANIPULATION

In this unit the students will be able to:

- Recall factorization of expressions of the following types.
  $ka + kb + kc$, $ac + ad + bc + bd$, $a^2 \pm 2ab + b^2$, $a^2 - b^2$, $a^2 \pm 2ab + b^2 - c^2$
- Factorize the expressions of the following types:
  $a^4 + a^2b^2 + b^4$ or $a^4 + 4b^4$, $x^2 + px + q$, $ax^2 + bx + c$,
  $(ax^2 + bx + c)(ax^2 + bx + d) + k$, $(x + a)(x + b)(x + c)(x + d) + k$
  $(x + a)(x + b)(x + c)(x + d) + kx^2$
  $a^3 + 3a^2b + 3ab^2 + b^3$, $a^3 - 3a^2b + 3ab^2 - b^3$, $a^3 \pm b^3$
- Find highest common factor (HCF) and least common multiple (LCM) of algebraic expressions.
- Use factor or division method to determine highest common factor and least common multiple.
- Know the relationship between HCF and LCM.
- Solve real life problems related to HCF and LCM.
- Use highest common factor and least common multiple to reduce fractional expressions involving $+, -, \times, \div$.
- Find square root of algebraic expression by factorization and division.

Ansel Adams (1902 – 1984) was a famous American photographer known for his style of detailed and focused photos that showed its subjects simply and directly. To take sharp and clear pictures, Adams had to focus the camera precisely. The distance from the object to the lens 'p' and the distance from the lens to the film 'q' must be calculated accurately to ensure that sharp image. The focal length of the lens is 'f'. The formula that relates these measurements is $\frac{1}{p}+\frac{1}{q}=\frac{1}{f}$. This formula involves addition of two algebraic fractions with the help of LCM.



4.1 Factorization of an Algebraic Expression

The process in which an algebraic expression can be expressed as the product of its factors is called its factorization. For example,

$$15x^2y = (5x)(3xy)$$

Here, $5x$ and $3xy$ are the factors of $15x^2y$.

Hence, the factorization process which converts expressions like $15x^2y$ into $(5x)(3xy)$ is essentially the opposite of the multiplication process.

Multiply

Polynomial	=	Factors
$15x^2y$		$(5x)(3xy)$

Factorize

In the previous grades, we have learnt about the factorization of polynomials of the following types:

(i)	$ka + kb + kc = k(a + b + c)$
(ii)	$ac + ad + bc + bd = (c + d)(a + b)$
(iii)	$a^2 \pm 2ab + b^2 = (a \quad b)^2$
(iv)	$a^2 - b^2 = (a + b)(a - b)$
(v)	$a^2 + 2ab + b^2 - c^2 = (a + b + c)(a + b - c)$
(vi)	$a^2 - 2ab + b^2 - c^2 = (a - b + c)(a - b - c)$

Let us learn some more about the factorization of polynomials.

Type-I → **Factorizing Expressions of the Forms**

(a) $a^4 + a^2b^2 + b^4$ **(b)** $a^4 + 4b^4$

(a) To factorize $a^4 + a^2b^2 + b^4$ or $a^4 + 4b^4$, we shall modify it and try to make it in appropriate form to utilize the previous results for its factorization.

Consider: $a^4 + a^2b^2 + b^4 = (a^2)^2 + a^2b^2 + (b^2)^2$

Here first and last terms are perfect squares but middle term is not twice the product of the square root of first and last term. So, we shall add and subtract a^2b^2 to make it twice.

i.e. (including zero term)

$$\begin{aligned} a^4 + a^2b^2 + b^4 &= (a^2)^2 + a^2b^2 + a^2b^2 + (b^2)^2 - a^2b^2 \\ &= (a^2)^2 + 2a^2b^2 + (b^2)^2 - a^2b^2 \\ &= (a^2 + b^2)^2 - (ab)^2 \\ &= (a^2 + b^2 + ab)(a^2 + b^2 - ab) \end{aligned}$$

So, $a^4 + a^2b^2 + b^4 = (a^2 + b^2 + ab)(a^2 + b^2 - ab)$

(b) Similarly, $a^4 + 64b^4 = (a^2)^2 + (8b^2)^2$

(zero term)

$$\begin{aligned} &= (a^2)^2 + 2(a^2)(8b^2) + (8b^2)^2 - 2(a^2)(8b^2) \\ &= (a^2 + 8b^2)^2 - 16a^2b^2 \\ &= (a^2 + 8b^2)^2 - (4ab)^2 \\ &= (a^2 + 8b^2 + 4ab)(a^2 + 8b^2 - 4ab) \end{aligned}$$

Example 1: Factorize the expression $8x^4 - 26x^2m^2 + 18m^4$.

Solution:

$$8x^4 - 26x^2m^2 + 18m^4$$
$$= 2(4x^4 - 13x^2m^2 + 9m^4)$$
$$= 2[(2x^2)^2 - 13x^2m^2 + (3m^2)^2]$$
$$= 2[(2x^2)^2 + 2(2x^2)(3m^2) + (3m^2)^2 - 2(2x^2)(3m^2) - 13x^2m^2]$$
$$= 2[(2x^2 + 3m^2)^2 - 12x^2m^2 - 13x^2m^2]$$
$$= 2[(2x^2 + 3m^2)^2 - 25x^2m^2]$$
$$= 2[(2x^2 + 3m^2)^2 - (5xm)^2]$$
$$= 2(2x^2 + 3m^2 + 5xm)(2x^2 + 3m^2 - 5xm)$$

> **Check Point**
> Can you factorize?
> $u^4 - u^2 + 1$

Type-II → **Factorization of a Trinomial of the Form**

(i) $x^2 + px + q$ **(ii)** $ax^2 + bx + c$

To factorize a trinomial of this form means to express the trinomial as the product of two binomials. To factorize such trinomials keep in mind the following steps:

(i) Make a list of all possible factors of 'product of extreme coefficients'.

(ii) Select a pair of factors among the list such that:

- their sum is equal to middle coefficient if extremes terms have same signs.
- their difference is equal to middle coefficient if extremes terms have opposite signs.

Example 2: Factorize the following polynomials.

(a) $y^2 - 7y + 12$ **(b)** $m^2 - 2m - 15$

Solution: (a)

$$y^2 - 7y + 12$$
$$= y^2 - 4y - 3y + 12$$
$$= y(y-4) - 3(y-4)$$
$$= (y-4)(y-3)$$

> Consider only negative factors when the middle term is negative and the coefficients of last term is positive.

So, $y^2 - 7y + 12 = (y-3)(y-4)$

product of –3 and –4 (12)

sum of –3 and –4 (–7)

(b)

$$m^2 - 2m - 15$$
$$= m^2 - 5m + 3m - 15$$
$$= m(m-5) + 3(m-5)$$
$$= (m-5)(m+3)$$

> Consider one positive and one negative factor when the coefficient of middle term and of last term are negative.

Example 3: Factorize $3x^2 + 22x - 16$ ← when middle term is positive & last term is negative.

Solution:

$$3x^2 + 22x - 16$$
$$= 3x^2 + 24x - 2x - 16$$
$$= (3x^2 + 24x) - (2x + 16)$$
$$= 3x(x+8) - 2(x+8)$$
$$= (x+8)(3x-2)$$

EXERCISE 4.1

Factorize the following polynomials.

1. $2x^2y^3 - 6x^2y^2 + 2xy^3$ **2.** $3nx - 3x - 3ny + 3y$

3. $18x^4 + 108x^2y^2 + 162y^4$ **4.** $(k + 2)^2 - 8(k + 2) + 16$

5. $9x^2 + 4 - 169y^2 - 12x$ **6.** $(x^2 - 1)(y + 1) - (y + 3)(x^2 - 1)$

7. $x^2 - 6ax + 9a^2 - 16b^2$ **8.** $1 - x^2 - 2xy - y^2$

9. Find a polynomial whose factorization is $(x + y - 2c)(x + 2c + y)$ by using an appropriate formula.

10. Show the expression $x^2 + 4y^2 - z^2 + 4xy$ as the difference of two squares.

11. Find the missing factor in the following.

(a) $(2y^2 - 3y - 27) = (y + 3)(\quad\quad)$ **(b)** $(5x^2 + 12x - 9) = (\quad\quad)(x + 3)$

Factorize the following expressions.

12. $x^4 + 4m^4$ **13.** $m^4 + m^2 + 1$ **14.** $3x^4 \quad 21x^3 + 24x^2$

15. $x^8 + x^4 + 1$ **16.** $4x^4 + 256y^4$ **17.** $12 - 7x + x^2$

18. $x^2 - 9x + 8$ **19.** $10z^2 - 29z + 10$ **20.** $-3y^2 + 13y - 4$

21. $x^2 - 21x + 90$ **22.** $x^2 + x - 2$ **23.** $3x^2 + 11x + 6$

24. $2x^2 - 5xy - 3y^2$ **25.** $8 + 6x - 5x^2$ **26.** $6 - 7x - 5x^2$

27. $2a^2 - 4a - 6$ **28.** $u^4 - 13u^2 + 36$ **29.** $y^4 - 12y^2 - 64$

Type-III → **Factorizing Expressions of the Forms**

(a) $(ax^2 + bx + c)(ax^2 + bx + d) + k$

(b) $(x + a)(x + b)(x + c)(x + d) + k$

(c) $(x + a)(x + b)(x + c)(x + d) + kx^2$

The process of factorizing expressions of above types will be explained in the following examples.

(a) $(ax^2 + bx + c)(ax^2 + bx + d) + k$

Example 4: Factorize: $(x^2 + 3x - 4)(x^2 + 3x + 5) + 8$

We observe here that first two terms inside both the parentheses are same. i.e.

$$(x^2 + 3x - 4 \quad (x^2 + 3x + 5) + 8$$

same

Let $x^2 + 3x = a$, then above expression will take the form

$$
\begin{aligned}
&= (a - 4)(a + 5) + 8 \\
&= a(a + 5) - 4(a + 5) + 8 \\
&= a^2 + 5a - 4a - 20 + 8 = a^2 + a - 12 \\
&= a^2 + 4a - 3a - 12 \\
&= (a^2 + 4a) - (3a + 12) \\
&= a(a + 4) - 3(a + 4) = (a + 4)(a - 3) \\
&= (x^2 + 3x + 4)(x^2 + 3x - 3) \text{ .....by putting the value of a.}
\end{aligned}
$$

(b) $(x+a)(x+b)(x+c)(x+d)+k$

To factorize such expressions, consider the the following examples.

Example 5a: Factorize: $(x+5)(x+3)(x+2)(x+6)-88$

Solution: $(x+5)(x+3)(x+2)(x+6)-88$ …… notice here, $5+3=2+6$

$$=[(x+5)(x+3)][(x+2)(x+6)]-88$$
$$=(x^2+8x+15)(x^2+8x+12)-88$$

Let $x^2+8x=a$, then above expression will take the form

$$=(a+15)(a+12)-88$$
$$=a^2+27a+180-88$$
$$=a^2+27a+92$$
$$=a^2+4a+23a+92$$
$$=(a^2+4a)+(23a+92)$$
$$=a(a+4)+23(a+4)=(a+4)(a+23)$$

$\therefore$ By back substitution

$$=(x^2+8x+4)(x^2+8x+23)$$

Example 5b: Factorize the expression

$$(z+1)(z-5)(z-9)(z-3)+44$$

Solution:

$$(z+1)(z-5)(z-9)(z-3)+44$$

Combine $(z+1)$ with $(z-9)$ and $(z-5)$ with $(z-3)$.

Re-arranging the given expression, we have

$$=(z+1)(z-9)(z-5)(z-3)+44$$
$$=(z^2-8z-9)(z^2-8z+15)+44$$

By putting $z^2-8z=x$ in the above expression,

it will take the form

$$=(x-9)(x+15)+44$$
$$=x^2-9x+15x-135+44$$
$$=x^2+6x-91$$
$$=x^2+13x-7x-91$$
$$=(x^2+13x)-(7x+91)$$
$$=x(x+13)-7(x+13)$$
$$=(x+13)(x-7)$$

Now replacing x by z^2-8z, we have

$$=(z^2-8z+13)(z^2-8z-7)$$

Check Point

While assuming same binomials equal to another variable, you must have to consider that variable which is not already present in the given expression.
Do you know why?

(c) $(x+a)(x+b)(x+c)(x+d)+kx^2$

Example 6: **Factorize the expression.**

$$(x+1)(x+2)(x+3)(x+6)-3x^2$$

Solution: $(x+1)(x+6)(x+2)(x+3)-3x^2$

$$=(x^2+6+7x)(x^2+6+5x)-3x^2$$

let $x^2+6=y$

$$=(y+7x)(y+5x)-3x^2$$
$$=y^2+5xy+7xy+35x^2-3x^2$$
$$=y^2+12xy+32x^2$$
$$=y^2+8xy+4xy+32x^2$$
$$=y(y+8x)+4x(y+8x)$$
$$=(y+8)(y+4x)$$

putting the value of y

$$=(x^2+6+8x)(x^2+6+4x)$$
$$=(x^2+8x+6)(x^2+4x+6)$$

Type-IV → **Factorizing expressions of the forms:**

(a) $a^3 + 3a^2b + 3ab^2 + b^3$
(b) $a^3 - 3a^2b + 3ab^2 - b^3$

We have studied in the previous unit that:

(i) $a^3 + 3a^2b + 3ab^2 + b^3 = (a + b)^3$
(ii) $a^3 - 3a^2b + 3ab^2 - b^3 = (a - b)^3$

The technique of factorization is elaborated through following examples.

Example 7: Factorize the following expressions.

(i) $27a^3 + 1 + 27a^2 + 9a$ (ii) $x^3 + \frac{3}{x} - \frac{1}{x^3} - 3x$

(iii) $4x^2(2x - 15) - 25(5 - 6x)$

Solution:

(i) $27a^3 + 1 + 27a^2 + 9a$

$= (3a)^3 + (1)^3 + 3(3a)^2(1) + 3(3a)(1)^2$
$= (3a)^3 + 3(3a)^2(1) + 3(3a)(1)^2 + (1)^3$
$= (3a + 1)^3$

(ii) $x^3 + \frac{3}{x} - \frac{1}{x^3} - 3x$

$= x^3 - 3x + \frac{3}{x} - \frac{1}{x^3}$

$= (x)^3 - 3(x)^2\left(\frac{1}{x}\right) + 3(x)\left(\frac{1}{x}\right)^2 - \left(\frac{1}{x}\right)^3$

$= \left(x - \frac{1}{x}\right)^3$

(iii) $4x^2(2x - 15) - 25(5 - 6x)$

$= 8x^3 - 60x^2 - 125 + 150x$
$= 8x^3 - 60x^2 + 150x - 125$
$= (2x)^3 - 3(2x)^2(5) + 3(2x)(5)^2 - (5)^3$
$= (2x - 5)^3$

Type-V → **Factorizing the sum and difference of two cubes**

(a) $a^3 + b^3$ (b) $a^3 - b^3$

We have studied in the previous unit that:

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2) \quad \text{and} \quad a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Example 8: Factorize: $x^3 + 27y^3$

Solution:

As, $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

So, $x^3 + 27y^3 = (x)^3 + (3y)^3 = (x + 3y)[(x)^2 - (x)(3y) + (3y)^2]$
$= (x + 3y)(x^2 - 3xy + 9y^2)$

Example 9: Factorize $x^6 - y^6$.

Solution: $x^6 - y^6 = (x^3)^2 - (y^3)^2$
$= (x^3 + y^3)(x^3 - y^3)$

$= (x+y)(x^2-xy+y^2)(x-y)(x^2+xy+y^2)$
$= (x+y)(x-y)(x^2-xy+y^2)(x^2+xy+y^2)$

Example 10: Factorize $8a^3 - 125b^3 - 2a + 5b$

Solution: $8a^3 - 125b^3 - 2a + 5b$

$= (2a)^3 - (5b)^3 - (2a - 5b)$
$= (2a - 5b)\ [(2a)^2 + (2a)(5b) + (5b)^2] - (2a - 5b)$
$= (2a - 5b)\ (4a^2 + 10ab + 25b^2) - (2a - 5b)$
$= (2a - 5b)\ (4a^2 + 10ab + 25b^2 - 1)$ ← factor out $(2a - 5b)$

EXERCISE 4.2

Factorize the following expressions completely.

1. $x^3 - 125$
2. $8x^3 + 1$
3. $3p^3q^3 - 81x^3$
4. $27 + 512x^3$
5. $t^6 - 64$
6. $x^6 + y^6$
7. $(2-x)^3 + (y-2)^3$
8. $64(x+y)^3 - z^3$
9. $27p^3 + 144pq^2 - 108p^2q - 64q^3$
10. $8p^3 + q^3 + 12p^2q + 6pq^2$
11. $125x^3 - y^3 - 75x^2y + 15xy^2$
12. $p^3 - 9p^2q + 27pq^2 - 27q^3$
13. $(2x^2 - 3x + 6)(2x^2 - 3x) - 55$
14. $(y^2 + 2y - 3)(y^2 + 2y + 11) + 48$
15. $y(y-1)(y-3)(y-4) + 2$
16. $(k+2)(k-3)(k+5)(k+10) + 375$
17. $(x-5)(x-6)(x+3)(x+2) + 12$
18. $(x+1)(x+2)(x-3)(x-6) - 21x^2$
19. $(x-2)(x-6)(x-3)(x-4) - 2x^2$
20. $(5-x)(2+x)(10-x)(1+x) - 7x^2$
21. The expression $a^6 + 729$ can be written in two ways as :
    (a) sum of two squares (b) sum of two cubes,
    which one will be used for factoring it and why? Also factorize the given expression.
22. Express $8 + 12t + 6t^2 + t^3$ as the product of three factors. Is each factor a binomial or a trinomial?

4.2 Highest Common Factor of Algebraic Expressions

As algebra is an extension of arithmetic, so we apply almost the same rules for finding HCF of two or more algebraic expressions (polynomials) as used in arithmetic.

4.2.1 Highest Common Factor by Factorization

A factor of a polynomial is an-other polynomial which divides it completely. The common factor of two or more polynomials is a polynomial which divides them exactly.

> The highest common factor of two or more than two polynomials is a highest degree polynomial which divides the given polynomials exactly.

(a) HCF of Monomial Expressions

To Find the HCF of Monomials:
I. Determine the HCF of numerical coefficients by prime factorization.
II. Determine the common variables and select their lowest power that appears in all monomials.
This will be the HCF of variables.

Example 11: Find the HCF of $18ab^3c$, $30a^2b^4c^3$, $24b^2c^5$.
Solution: First we find the HCF of numerical coefficients 18, 30 and 24 as

$$18 = 2 \times 3^2$$
$$30 = 2 \times 3 \times 5$$
$$24 = 2^3 \times 3$$

$\therefore$ HCF of 18, 30 and 24 = $2 \times 3 = 6$
HCF of b^2, b^3 and b^4 is b^2. ⟵ least common power of b
HCF of c, c^3 and c^5 is c. ⟵ least common power of c
Hence, required HCF = $6b^2c$

Food for Thought
What is HCF of $7x^2$ and $5y^2$?

(b) HCF of Compound Polynomial Expressions

To Find the HCF of Compound Expressions:
I. Write each expression in complete factored form. Repeated factors should be expressed as powers.
II. Select the least power of each common factor.
III. The highest common factor (HCF) is the product of results of step-II.

Example 12: Find the HCF of $2m^2 - 2mn$, $4m^4 - 4m^2n^2$ and $2m^3 - 4m^2n + 2mn^2$.
Solution: First factor each polynomial expression completely as

$$2m^2 - 2mn = 2m(m-n)$$
$$4m^4 - 4m^2n^2 = 4m^2(m^2 - n^2)$$
$$= 2^2m^2(m+n)(m-n)$$
$$2m^3 - 4m^2n + 2mn^2 = 2m(m^2 - 2mn + n^2)$$
$$= 2m(m-n)^2$$

Common factors with least power are 2, m, m – n

$\therefore$ Required HCF = $2m(m-n)$

Example 13: Find the HCF of the following.
$$ax^2 + 7ax + 12a,\ ax^2 - 5ax - 24a,\ 2ax^2 + 5ax - 3a$$
Solution: $ax^2 + 7ax + 12a = a(x^2 + 7x + 12)$
$$= a(x^2 + 4x + 3x + 12)$$
$$= a[x(x+4) + 3(x+4)]$$
$$= a(x+3)(x+4)$$
$ax^2 - 5ax - 24a = a(x^2 - 5x - 24)$
$$= a(x^2 - 8x + 3x - 24)$$
$$= a[x(x-8) + 3(x-8)]$$

$= a(x+3)(x-8)$

$2ax^2 + 5ax - 3a = a(2x^2 + 5x - 3)$

$= a(2x^2 + 6x - x - 3)$

$= a[2x(x+3) - 1(x+3)]$

$= a(2x-1)(x+3)$

$\therefore$ Required HCF = product of all common factors with least power

$= a(x+3)$

Food for Thought

What would be the HCF of a^3-b^3 and a^3+b^3?

EXERCISE 4.3

1.(a) Find the HCF of the following monomials by completing the table.

Monomials	HCF of numerical coefficients	HCF of 'p'	HCF of 'q'	HCF of 'r'	Required HCF
$16p^3q$, $9pq^2r$					
$10p^3q^2r$, $5p^2qr$, $15p^2qr^2$					
$14p^4qr^4$, $28p^3qr^2$, $7p^2qr^2$, $21p^2q^2r^4$					

(b) If all common factors with least power of three unknown polynomials are 2^2, 3, pq and $(p+q)^2$ then what would be their HCF?

(c) Write any two polynomials of your choice having HCF as 1.

(d) The only common factor of two polynomials is $m-n$ and the only uncommon factor is m^2+n^2. Can you guess the unknown polynomials?

(e) Can you guess HCF of two polynomials x^3+5x+1 and $1+5x+x^3$ without any procedure?

Find the HCF of the following by factorization.

2. $(x+y)^2$, x^2-y^2
3. $(a-b)^3$, $a^2-2ab+b^2$
4. a^3b-ab^3, $a^5b^2-a^2b^5$
5. x^2-49, $x^2-4x-21$
6. $12x^2+x-1$, $15x^2+8x+1$
7. $c^2x^2-d^2$, $acx^2-bcx+adx-bd$
8. m^2-n^2, m^4-n^4, m^6-n^6
9. $ax^2+2a^2x+a^3$, $2ax^2-4a^2x-6a^3$, $3(ax+a^2)^2$

4.2.2 Highest Common Factor by Division

Sometimes, it is difficult to factorize the given polynomials completely. In such cases, we adopt division method to find the HCF of these polynomials.
We will explain the procedure with the help of the following example.

Example 14: Find the HCF of two polynomials x^3+x^2-5x+3 and x^2+3x.

Solution: The following steps will be followed for finding HCF.

Step-I: Arrange the given polynomials in descending order w.r.t. the variables.

In this case, polynomials are already in descending order.

Step-II: Consider the higher degree polynomial as dividend and lower degree polynomial as divisor and start division process.

Step-III: Since degree of remainder has become smaller than the degree of divisor, so remainder will be taken as divisor and divisor will be considered as dividend.

$$x^2+3x\,\big)\,x^3+x^2-5x+3\,\big(\,x-2$$
$$\underline{\pm x^3 \pm 3x^2}$$
$$-2x^2-5x+3$$
$$\underline{\mp 2x^2 \mp 6x}$$
$$x+3\,\big)\,x^2+3x\,\big(\,x$$
$$\underline{\pm x^2 \pm 3x}$$
$$0$$

Divisor

Continue the same process until we get 0 remainder. As $x+3$ is the last divisor which gives remainder as '0'. Therefore, required HCF $= x+3$

Example 15: Find the HCF of $12+16a+7a^2+a^3$ and $13a+5a^2+14+a^3$.

Solution: First arrange the given polynomials in descending order w.r.t. the variable. i.e. $a^3+7a^2+16a+12$ and $a^3+5a^2+13a+14$. Since degree of both the polynomials is same, so any one can be taken as a divisor or dividend.

$$a^3+5a^2+13a+14\,\big)\,a^3+7a^2+16a+12\,\big(\,1$$
$$\underline{\pm a^3 \pm 5a^2 \pm 13a \pm 14}$$
$$2a^2+3a-2\,\big)\,a^3+5a^2+13a+14\,\big(\,a+7$$
$$\underline{\times 2}$$
$$2a^3+10a^2+26a+28$$
$$\underline{\pm 2a^3 \pm 3a^2 \mp 2a}$$
$$7a^2+28a+28$$
$$\underline{\times 2}$$
$$14a^2+56a+56$$
$$\underline{\pm 14a^2 \pm 21a \mp 14}$$
$$35a+70 = 35(a+2)$$

Since first term of the divisor is $2a^2$ and first term of the dividend is a^3. Therefore, for convenience, first we multiply the dividend by 2.

Again first we will multiply the remainder by 2.

Since 35 is not a factor of any polynomial so we can neglect it for convenience.

Now divide $2a^2+3a-2$ by $a+2$.

$$a+2\,\big)\,2a^2+3a-2\,\big(\,2a-1$$
$$\underline{\pm 2a^2 + 4a}$$
$$-a-2$$
$$\underline{\mp a \mp 2}$$
$$0$$

$\therefore$ Required HCF $= a+2$

Key Fact

HCF of two polynomials P and Q will be the same as that of mP and nQ (where m and n are non zero constants). So in the process of finding the HCF, we can multiply or divide any divisor or dividend by any suitable number according to our requirement, but we cannot multiply or divide it by a variable.

4.3 Least Common Multiple of Algebraic Expressions

The least common multiple of two or more polynomials is a polynomial of lowest degree that contains the factors of each polynomial.

4.3.1 Least Common Multiple by Factorization

The process of determining the LCM is almost identical to that for determining the HCF. Prime factorization is also useful to determine the LCM of two or more polynomials. The LCM is obtained by taking product of all factors (common and uncommon) with highest power.

(a) LCM of Monomial Expressions

To determine the LCM of two or more monomials, find LCM of the numerical coefficients and LCM of each variable. Then find their product for the required LCM.
To understand this concept, consider the two monomials $45x^2y$ and $60x^3y^2z$.
First find the LCM of numerical coefficients 45 and 60 which is 180.
Now consider variables x, y and z.

LCM of x^2 and x^3 = highest power of x appearing in any monomial = x^3
LCM of y and y^2 = highest power of y appearing in any monomial = y^2
LCM of z = highest power of z appearing in any monomial = z

Thus, required LCM = product of LCM of coefficients and LCM of each variable.
$= 180x^3y^2z$

Example 16: Find LCM of $18a^3b^4c^5$, $60a^3b^4c^6$ and $42a^4b^3$.

Solution: First we find the LCM of 18, 60 and 42 by prime factorization as

$$18 = 2\times 3^2,\quad 60 = 2^2\times 3\times 5,\quad 42 = 2\times 3\times 7$$

$\therefore$ LCM of 18, 60 and 42 = product of all factors with highest power
$= 2^2\times 3^2\times 5\times 7 = 1260$

Now we find LCM of each variable as

LCM of a^3 and a^4 = a^4 (highest power of a)
LCM of b^3 and b^4 = b^4 (highest power of b)
LCM of c^5 and c^6 = c^6 (highest power of c)
$\therefore$ Required LCM = $1260a^4b^4c^6$

> **Thinking Zone**
>
> Haleema has $60(x+1)$ english story books, $45(x+1)$ math fun books and $75(x+1)$ science fun books. She wants to put all books in groups of same number. What do you think can be the biggest number of books to be put in each group.

(b) LCM of Compound Polynomial Expressions

To understand the procedure, let us consider the following examples.

Example 17: Find LCM of $x^4y^3 - x^3y^4$ and $x^3y - xy^3$

Solution: First we factorize each polynomial completely.

$$x^4y^3 - x^3y^4 = x^3y^3(x-y)$$

$$x^3y - xy^3 = xy(x^2-y^2) = xy(x-y)(x+y)$$

We choose every factor with highest power as: x^3y^3, $x-y$ and $x+y$.

Then their product is $x^3y^3(x+y)(x-y)$

$\therefore$ Required LCM $= x^3y^3(x+y)(x-y)$
$= x^3y^3(x^2-y^2)$
$= x^5y^3 - x^3y^5$

To find the LCM of Compound Expressions:

I. Write each polynomial in complete factored form. Repeated factors should be expressed as powers.

II. Select the highest power of every factor that appears.

III. The least common multiple is the product of the results of step II.

Example 18: Find the LCM of $2x^2 - 12xy + 16y^2$, $x^2 - 6xy + 8y^2$ and $3x^2 - 12y^2$.

Solution:

$$\begin{aligned} 2x^2 - 12xy + 16y^2 &= 2(x^2 - 6xy + 8y^2) \\ &= 2(x-4y)(x-2y) \\ x^2 - 6xy + 8y^2 &= (x-4y)(x-2y) \\ 3x^2 - 12y^2 &= 3(x^2 - 4y^2) \\ &= 3(x+2y)(x-2y) \end{aligned}$$

All factors with highest powers are $2,\ 3,\ x-4y,\ x-2y,\ x+2y$

$$\begin{aligned} \therefore \text{ Required LCM} &= 2 \times 3 \times (x-4y)(x-2y)(x+2y) \\ &= 6(x-4y)(x^2-4y^2) \end{aligned}$$

Note: Same pattern will be followed for more than three polynomials.

Key Fact

- The key words in the two processes of HCF and LCM are:
  HCF ⟶ lowest power and common factor.
  LCM ⟶ highest power and every factor.
- LCM of two or more polynomials is a lowest degree polynomial that is exactly divisible by each of the given polynomials.

EXERCISE 4.4

1. Give quick answers to these questions without doing any procedure.
   If HCF of two polynomials $x^3 + 5x^2 + 6x$ and $x^3 + 9x^2 + 14x$ is obtained as $x^2 + 2x$, then:

(i) What would be the HCF of $5(x^3 + 5x^2 + 6x)$ and $x^3 + 9x^2 + 14x$?

(ii) What would be the HCF of $x^3 + 5x^2 + 6x$ and $2(x^3 + 9x^2 + 14x)$?

(iii) What would be the HCF of $3(x^3 + 5x^2 + 6x)$ and $7(x^3 + 9x^2 + 14x)$?

(iv) What would be the HCF of $15(x^3 + 5x^2 + 6x)$ and $25(x^3 + 9x^2 + 14x)$?

(v) Does HCF of the given polynomials will remain unchanged if both are multiplied by x?

2. Find the LCM of the following monomials by completing the table.

Monomials	LCM of numerical coefficients	LCM of 'x'	LCM of 'y'	LCM of 'z'	Required LCM
(i) $8x^6y,\ 4x^2yz$					
(ii) $12x^2y^4z,\ 24x^3z$					
(iii) $18x^3z,\ 9xy^2z,\ 6x^6yz^3$					
(iv) $xyz^3,\ z^5y^3x,\ 28x^3y^5z$					

Find the HCF of the following by division method.

3. $a^2 + a - 2,\ a^3 + 2a^2 + a + 2$

4. $x^3 + 2x^2 - 4x - 8,\ 2x^3 + 7x^2 + 4x - 4$

5. $2x^3 + x^2 - x - 2,\ 3x^3 - x^2 + x - 3$

6. $3 + 2p^4 + 5p^2,\ 5p + 5p^3 + 3 + 3p^2$

7. $24x^4 - 2x^3 - 60x^2 - 32x,\ 18x^4 - 6x^3 - 39x^2 - 18x$

8. $2x^3 + 6x^2 + x + 3,\ 3x^3 + 9x^2 - 2x - 6,\ x^3 + 3x^2 + 2x + 6$

Find the LCM of the following expressions.

9. $9a^2b - b,\ 6a^2 + 2a$

10. $p^3q - pq^3,\ p^5q^2 - p^2q^5$

11. $4x^2y - y,\ 2x^2 + x$

12. $x^2 - x - 6,\ x^2 + x - 2,\ x^2 - 4x + 3$

13. $m^6 - 1,\ m^4 - 1,\ m^3 - 1$

14. $x^3 + 2x^2 - x - 2,\ x^2 - x - 2,\ x^2 - 4$

15. $x^2 + x - 20,\ x^2 - 10x + 24,\ x^2 - x - 30$

4.3.2 Relation between HCF and LCM

Consider two polynomials $P = a^2 - b^2$ and $Q = a^2 - 2ab + b^2$. For their LCM and HCF, first factorize them as

$$P = a^2 - b^2 = (a+b)(a-b)$$
$$Q = a^2 - 2ab + b^2 = (a-b)^2$$

$\therefore$ HCF of P and Q $= a - b$

and LCM of P and Q $= (a+b)(a-b)^2$

Now HCF × LCM $= (a-b)(a+b)(a-b)^2$

$= (a^2 - b^2)(a-b)^2$

$= (a^2 - b^2)(a^2 - 2ab + b^2)$ .... (i)

Also, product of P and Q $= (a^2 - b^2)(a^2 - 2ab + b^2)$ .... (ii)

Thus, LCM × HCF = P × Q .... (1)

Hence, it can be generalized that:

product of their HCF and LCM = product of given polynomials

4.3.3 Finding of Least Common Multiple by Division

Sometimes, it is much difficult to find the LCM of given polynomials P and Q by factorization method. Then in that case, we can find the LCM by division as follows. From the relation (1) we have,

$$\text{LCM} = \frac{P \times Q}{HCF} = \frac{P}{HCF} \times Q = \frac{Q}{HCF} \times P$$

Procedure is illustrated through examples.

Example 19: Find the LCM of $P = 10x^4 + 3x^3 + 8$ and $Q = 8x^4 + 3x + 10$

Solution: First we will find their HCF as

$$\begin{array}{r}
5 \\
8x^4 + 3x + 10 \overline{)\, 10x^4 + 3x^3 + 8} \\
\times 4 \quad \leftarrow \text{multiplying dividend by 4} \\
\hline
40x^4 + 12x^3 + 32 \\
\underset{-}{+}40x^4 \quad \underset{-}{+}50 \; \underset{-}{+}15x \\
\hline
12x^3 - 18 - 15x
\end{array}$$

where $(12x^3 - 18 - 15x) \div 3 = 4x^3 - 6 - 5x = 4x^3 - 5x - 6$

Now, we will divide $8x^4 + 3x + 10$ by $4x^3 - 5x - 6$

$$\begin{array}{r}
2x \\
4x^3 - 5x - 6 \overline{)\, 8x^4 + 3x + 10} \\
\underset{-}{+}8x^4 \underset{+}{-}12x \quad \underset{+}{-}10x^2 \\
\hline
15x + 10 + 10x^2 = 10x^2 + 15x + 10
\end{array}$$

Again, $(10x^2 + 15x + 10) \div 5 = 2x^2 + 3x + 2$

$$\begin{array}{r}
2x - 3 \\
2x^2 + 3x + 2 \overline{)\, 4x^3 - 5x - 6} \\
\underset{-}{+}4x^3 \underset{-}{+}4x \quad \underset{-}{+}6x^2 \\
\hline
-6x^2 - 9x - 6 \\
\underset{+}{-}6x^2 \underset{+}{-}9x \underset{+}{-}6 \\
\hline
0
\end{array}$$

$\therefore$ HCF is $2x^2 + 3x + 2$.

Now, we obtain the LCM using the following relation.

$$\text{LCM} = \frac{P \times Q}{HCF} = \frac{(10x^4 + 3x^3 + 8)(8x^4 + 3x + 10)}{2x^2 + 3x + 2}$$

(Since, the HCF of two polynomials divides both of them exactly. So divide any polynomial of numerator by denominator.)

$$\begin{array}{r}
5x^2 - 6x + 4 \\
2x^2 + 3x + 2 \overline{)\, 10x^4 + 3x^3 + 0x^2 + 0x + 8} \\
\underset{-}{+}10x^4 \underset{-}{+}15x^3 \underset{-}{+}10x^2 \\
\hline
-12x^3 - 10x^2 + 0x + 8 \\
\underset{+}{-}12x^3 \underset{+}{-}18x^2 \underset{+}{-}12x \\
\hline
8x^2 + 12x + 8 \\
\underset{-}{+}8x^2 \underset{-}{+}12x \underset{-}{+}8 \\
\hline
0
\end{array}$$

> **Memory Plus**
>
> HCF is not affected by multiplying or dividing any polynomial with any number during the process of finding HCF.

Hence, required LCM $= (5x^2 - 6x + 4)(8x^4 + 3x + 10)$

Example 20: Find the second polynomial Q when first polynomial $P = x^2 - 5x + 6$,
HCF $= x - 3$ and LCM $= x^3 - 9x^2 + 26x - 24$

Solution: $P = x^2 - 5x + 6, \quad Q = ?$

LCM of P and Q $= x^3 - 9x^2 + 26x - 24$

HCF $= x - 3$

$$Q = \frac{HCF \times LCM}{P}$$

$$Q = \frac{(x-3)(x^3 - 9x^2 + 26x - 24)}{x^2 - 5x + 6}$$

$$Q = \frac{(x-3)(x^3 - 9x^2 + 26x - 24)}{(x-3)(x-2)}$$

$$Q = \frac{x^3 - 9x^2 + 26x - 24}{x-2} = x^2 - 7x + 12$$

$$\begin{array}{r|l} & x^2 - 7x + 12 \\ \hline x-2 & x^3 - 9x^2 + 26x - 24 \\ & \underline{\pm x^3 \mp 2x^2} \\ & \quad -7x^2 + 26x - 24 \\ & \quad \underline{\mp 7x^2 \pm 14x} \\ & \qquad 12x - 24 \\ & \qquad \underline{\pm 12x \mp 24} \\ & \qquad\qquad 0 \end{array}$$

EXERCISE 4.5

1. Find the HCF and LCM of the following.
   i. $16 - 4x^2, \ x^2 + x - 6$
   ii. $a^4 - a^3 - a + 1, \ a^4 + a^2 + 1$
   iii. $x^3 + 2x^2 - 3x, \ 2x^3 + 5x^2 - 3x$
2. If HCF and LCM of two polynomials are $x - 7$ and $x^3 - 10x^2 + 11x + 70$ respectively. Then find product of two polynomials.
3. Product of two polynomials is $x^4 + 3x^3 - 12x^2 - 20x + 48$ and their HCF is $x - 2$. Find their LCM.
4. The product of two polynomials is $y^4 + 6y^3 - 3y^2 - 56y - 48$ and their LCM is $y^3 + 2y^2 - 11y - 12$. Find their HCF.
5. Find the second polynomial when,
   First polynomial $= x^4 + x^3 + x + 1$, HCF $= x + 1$ and LCM $= (x^3 + 1)(x^4 + x^3 - x - 1)$
6. Find the LCM of polynomials $4x^3 - 10x^2 + 4x + 2$ and $3x^4 - 2x^3 - 3x + 2$ if their HCF is $x - 1$.

4.4 Basic Operations on Algebraic Fractions

In this topic addition, subtraction, multiplication and division of algebraic fractions will be discussed.

4.4.1 Algebraic Fractions

Algebraic expressions of the type $\dfrac{x^2 + 2x + 1}{2x^2 - 5x - 3}, \dfrac{-1 - t}{\frac{1}{t} + 1}, \dfrac{p^2}{qp^2}$

are called algebraic fractions.

4.4.2 Multiplication of Algebraic Fractions

We multiply two or more algebraic fractions in the same way as common fractions in arithmetic.

If $\frac{R}{T}$ and $\frac{S}{U}$ are two algebraic fractions, where $T \neq 0, U \neq 0$.

Then, $$\frac{R}{T} \times \frac{S}{U} = \frac{RS}{TU}$$

In general, before finding the product of two or more algebraic fractions, we factorize the numerator and denominator of each fraction, if possible, and divide out all the common factors, to get the most reduced form of the resulting fraction.

Key Fact

In the process of multiplication, product of all the common factors being cancelled is in fact HCF of the expressions present in numerator and denominator.

Example 21: Find the indicated product.

$$\frac{x^2-1}{x^2+4x+4} \times \frac{x+2}{x^2+2x-3}$$

Solution:

$$\frac{x^2-1}{x^2+4x+4} \times \frac{x+2}{x^2+2x-3}$$

$$= \frac{(x+1)(x-1)}{(x+2)(x+2)} \times \frac{x+2}{(x-1)(x+3)}$$

$$= \frac{x+1}{x+2} \times \frac{1}{x+3}$$

$$= \frac{x+1}{(x+2)(x+3)} = \frac{x+1}{x^2+5x+6}$$

Pointer to Ponder

The key to success in simplifying an algebraic fraction lies in your ability to factor the polynomials.

4.4.3 Division of Algebraic Fractions

If $\frac{S}{U}$ and $\frac{T}{Q}$ are two algebraic fractions, then

$$\frac{S}{U} \div \frac{T}{Q} = \frac{S}{U} \times \frac{Q}{T} = \frac{SQ}{UT}, \text{ where } U \neq 0, Q \neq 0 \text{ and } T \neq 0$$

To divide one fraction by another, invert the divisor and proceed as in multiplication because 'to divide' by a fraction means 'to multiply' by its reciprocal.

Example 22: Find the indicated operation.

$$\frac{r^2 - s^2}{r} \div \frac{r - s}{s}$$

Solution: $= \dfrac{r^2 - s^2}{r} \div \dfrac{r - s}{s}$

$$= \frac{(r+s)(r-s)}{r} \times \frac{s}{r-s} \quad = \quad \frac{r+s}{r} \times \frac{s}{1}$$

$$= \frac{s(r+s)}{r} \quad = \quad \frac{sr + s^2}{r}$$

Example 23: Simplify: $\dfrac{3x^3 + 6x^2}{2x^2 + x - 6} \div (6x^2 - 15x)$

Solution: $\dfrac{3x^3 + 6x^2}{2x^2 + x - 6} \div (6x^2 - 15x)$

$$= \frac{3x^3 + 6x^2}{2x^2 + x - 6} \times \frac{1}{6x^2 - 15x}$$

$$= \frac{3x^2(x+2)}{(x+2)(2x-3)} \times \frac{1}{3x(2x-5)}$$

$$= \frac{x}{(2x-3)(2x-5)} \quad = \quad \frac{x}{4x^2 - 16x + 15}$$

4.4.4 Addition and Subtraction of Algebraic Fractions

Algebraic fractions are added or subtracted just like arithmetic fractions i.e., by manipulating LCM of their denominators.

Example 24: Simplify the following.

$$\frac{x^2 + 2x + 2}{2x} + \frac{(-2x - 2)}{2x}$$

Solution: $\dfrac{x^2 + 2x + 2}{2x} + \dfrac{(-2x - 2)}{2x}$

$$= \frac{x^2 + 2x + 2 + (-2x - 2)}{2x}$$

$$= \frac{x^2 + 2x + 2 - 2x - 2}{2x}$$

$$= \frac{x^2}{2x} \quad = \quad \frac{x}{2}$$

Enlighten Yourself

Algebraic fractions with same denominators are called like fractions e.g., $\dfrac{a \ \ -2}{a+b}, \dfrac{b^2 - 2}{a+b}$

Fractions with different denominators are called unlike fractions e.g.,

$\dfrac{x^2 - y}{ax + b}, \dfrac{y^2 - x}{cx + d}$

Example 25: Perform the indicated operations.

$$\frac{-2x}{x+3}+\frac{3}{3-x}-\frac{8x-12}{x^2-9}$$

Solution: $\frac{-2x}{x+3}+\frac{3}{3-x}-\frac{8x-12}{x^2-9}$

$$=\frac{-2x}{x+3}+\frac{-3}{x-3}-\frac{8x-12}{(x+3)(x-3)} \quad \longleftarrow \quad \because 3-x=-(x-3)$$

$$=\frac{-2x(x-3)+(-3)(x+3)-(8x-12)}{(x+3)(x-3)} \quad \longleftarrow \quad \because \text{LCM is } (x+3)(x-3)$$

$$=\frac{-2x^2+6x-3x-9-8x+12}{(x+3)(x-3)} \quad \longleftarrow \quad \text{simplify the parentheses}$$

$$=\frac{-2x^2-5x+3}{(x+3)(x-3)} \quad \longleftarrow \quad \text{combine like terms}$$

$$=\frac{-1(2x^2+5x-3)}{(x+3)(x-3)} \quad \longleftarrow \quad \text{negative sign is taken as common}$$

$$=\frac{-1(2x-1)(x+3)}{(x+3)(x-3)}$$

$$=\frac{-(2x-1)}{(x-3)}$$

$$=-\frac{2x-1}{x-3} \quad \longleftarrow \quad \text{reduced form}$$

4.4.5 Algebraic Fractions with Combined Operations

When two or more operations occur in any algebraic expression then the rule for order of operations (DMAS) must be followed.

Example 26: Simplify.

$$\frac{3u^2+5uv+2v^2}{u^2+5uv+6v^2}+\frac{2u+6v}{u^2-4v^2}\div\frac{u^2+6uv+9v^2}{u^2v-2uv^2}$$

Solution: Division is performed before addition while simplifying an expression. Therefore, first we will simplify fractions having '÷' sign.

$$\frac{3u^2+5uv+2v^2}{u^2+5uv+6v^2}+\frac{2u+6v}{u^2-4v^2}\div\frac{u^2+6uv+9v^2}{u^2v-2uv^2}$$

$$=\frac{3u^2+5uv+2v^2}{u^2+5uv+6v^2}+\frac{2u+6v}{u^2-4v^2}\times\frac{u^2v-2uv^2}{u^2+6uv+9v^2}$$

$$=\frac{3u^2+5uv+2v^2}{u^2+5uv+6v^2}+\frac{2(u+3v)}{(u+2v)(u-2v)}\times\frac{uv(u-2v)}{(u+3v)^2}$$

$$=\frac{3u^2+5uv+2v^2}{u^2+5uv+6v^2}+\frac{2uv}{(u+2v)(u+3v)}$$

$$= \frac{3u^2 + 5uv + 2v^2}{(u + 2v)(u + 3v)} + \frac{2uv}{(u + 2v)(u + 3v)}$$

$$= \frac{3u^2 + 5uv + 2v^2 + 2uv}{(u + 2v)(u + 3v)}$$

$$= \frac{3u^2 + 7uv + 2v^2}{(u + 2v)(u + 3v)}$$

$$= \frac{(3u + v)(u + 2v)}{(u + 2v)(u + 3v)} = \frac{3u + v}{u + 3v}$$

EXERCISE 4.6

1. Answer these without calculations.

i. Product of what algebraic fraction and $x^3 + 7x - 8$ is 1?

ii. Which algebraic fraction divided by $\frac{x^2}{x^2 + y^2}$ gives 1?

iii. Sum of what algebraic fraction and $\frac{m}{m^2 + n^2}$ is $\frac{m + n}{m^2 + n^2}$?

iv. What is the product of an algebraic fraction and its reciprocal?

Simplify the following (where all the expressions in the denominator are non-zero).

2. $\frac{14x^2 - 7x}{12x^3 + 24x^2} \times \frac{x^2 + 2x}{2x - 1}$

3. $\frac{a^2b^2 + 3ab}{4a^2 - 1} \times \frac{2a + 1}{ab + 3}$

4. $\frac{a - b}{a^2 + ab} \times \frac{a^4 - b^4}{a^2 - 2ab + b^2} \times \frac{a}{a^2 + b^2}$

5. $\frac{6x^2y^2}{x^2 - y^2} \div \frac{3xy}{x + y}$

6. $\frac{8a^3 - 1}{4a^3 + 2a^2} \div \frac{6a^2 - 13a + 5}{15a - 25} \times \frac{2a^4 + a^3}{15a^2}$

7. $\frac{x^2 - 8x - 9}{x^2 - 17x + 72} \times \frac{x^2 - 25}{x^2 - 1} \div \frac{x^2 + 4x - 5}{x^2 - 9x + 8}$

8. $\frac{1}{2x - 3y} - \frac{x + y}{4x^2 - 9y^2}$

9. $\frac{1}{x(x - y)} + \frac{1}{y(x + y)}$

10. $\frac{5x + 5}{3(2x - 1)} + \frac{6 - 2x}{2(1 - 2x)}$

11. $\frac{2a}{2a - 3} - \frac{5}{6a + 9} - \frac{4(3a + 2)}{3(4a^2 - 9)}$

12. $\frac{5}{5 + x - 18x^2} - \frac{2}{2 + 5x + 2x^2}$

13. $\frac{1 - p^2}{1 + q} \times \frac{1 - q^2}{p + p^2} \times \left(1 + \frac{p}{1 - p}\right)$

4.5 Square Root of Algebraic Expressions

The square root of an algebraic expression is defined as one of its equal factors e.g., $x+y$ is the square root of $x^2+2xy+y^2$ because,

$$x^2+2xy+y^2=(x+y)^2$$

The square root of an algebraic expression P is another algebraic expression Q which, when squared, gives P. Thus, if $P=(-Q)^2$ then Q and – Q both are square roots of P.

Square root of an algebraic expression can be obtained in two ways.

i. Factorization Method ii. Division Method

4.5.1 Square Root by Factorization Method

In this method, before applying the square root, given expression is written in the form of a complete square. For example, to find the square root of $4a^2+12ab+9b^2$, first we will convert the given expression into a complete square as follows:

$$\begin{aligned}4a^2+12ab+9b^2 &= (2a)^2+2(2a)(3b)+(3b)^2\\ &= (2a+3b)^2\\ &= [\pm\ (2a+3b)]^2\end{aligned}$$

Now, applying square root on both sides, we get.

$$\sqrt{4a^2+12ab+9b^2}=\pm(2a+3b)$$

Key Fact

The square root of an algebraic expression consists of two expressions, which are additive inverses of each other.

Example 27: Find the square root of $\left(a+\frac{1}{a}\right)^2-4\left(a-\frac{1}{a}\right)$.

Solution: To find square root of such type of expressions, we can adopt two methods.

Method-I

$$\begin{aligned}\left(a+\frac{1}{a}\right)^2-4\left(a-\frac{1}{a}\right) &= a^2+2(a)\left(\frac{1}{a}\right)+\frac{1}{a^2}-4\left(a-\frac{1}{a}\right)\\ &= a^2+2+\frac{1}{a^2}-4\left(a-\frac{1}{a}\right)\\ &= a^2-2+\frac{1}{a^2}-4\left(a-\frac{1}{a}\right)+2+2\\ &= \left(a-\frac{1}{a}\right)^2-4\left(a-\frac{1}{a}\right)+4\\ &= \left(a-\frac{1}{a}\right)^2-2\left(a-\frac{1}{a}\right)(2)+(2)^2\\ &= \left(a-\frac{1}{a}-2\right)^2\end{aligned}$$

$$\sqrt{\left(a+\frac{1}{a}\right)^2-4\left(a-\frac{1}{a}\right)} = \pm\left(a-\frac{1}{a}-2\right)$$ ....(applying square root)

Method-II

$$\left(a+\frac{1}{a}\right)^2-4\left(a-\frac{1}{a}\right) = a^2+2+\frac{1}{a^2}-4\left(a-\frac{1}{a}\right)$$

$$= \left(a^2+\frac{1}{a^2}\right)+2-4\left(a-\frac{1}{a}\right) \quad \text{........ (i)}$$

Let, $a-\frac{1}{a}=x$ .... (a)

Then, $\left(a-\frac{1}{a}\right)^2=x^2$

or $a^2-2+\frac{1}{a^2}=x^2$

or $a^2+\frac{1}{a^2}=x^2+2$ .... (b)

Substituting values from equations (a) and (b) in equation (i)

$$\left(a+\frac{1}{a}\right)^2-4\left(a-\frac{1}{a}\right) = (x^2+2)+2-4x$$

$$= x^2-4x+4$$

$$= (x-2)^2 \quad \text{........ (ii)}$$

Now, replace x by $a-\frac{1}{a}$ in equation (ii)

$$\left(a+\frac{1}{a}\right)^2-4\left(a-\frac{1}{a}\right)=\left(a-\frac{1}{a}-2\right)^2$$

Applying square root on both sides, we get

$$\sqrt{\left(a+\frac{1}{a}\right)^2-4\left(a-\frac{1}{a}\right)} = \pm\left(a-\frac{1}{a}-2\right).$$

Example 28: Find the square root of $(2a+1)(2a+3)(2a+5)(2a+7)+16$.

Solution:

$$(2a+1)(2a+3)(2a+5)(2a+7)+16 = [(2a+1)(2a+7)][(2a+3)(2a+5)]+16$$

$$=(4a^2+16a+7)(4a^2+16a+15)+16$$

$$=(4a^2+16a+7)(4a^2+16a+7+8)+16$$

$$(2a+1)(2a+3)(2a+5)(2a+7)+16=x(x+8)+16 \quad \text{.....(put } 4a^2+16a+7=x)$$

$$=x^2+8x+16$$

$$=x^2+2(x)(4)+(4)^2 = (x+4)^2$$

Now replace x by $4a^2+16a+7$

$$(2a+1)(2a+3)(2a+5)(2a+7)+16 = (4a^2+16a+7+4)^2 = (4a^2+16a+11)^2$$

Applying square root on both sides, we get

$$\sqrt{(2a+1)(2a+3)(2a+5)(2a+7)+16} = \pm(4a^2+16a+11)$$

4.5.2 Square Root by Division Method

To understand this method, consider the following example.

Example 29: Find the square root of $9x^2-42xy+49y^2$ by division method.

Solution:

$$
\begin{array}{l|l}
 & \quad 3x-7y \leftarrow \text{root} \\ \hline
\left[\sqrt{9x^2}=3x\right] \rightarrow \quad 3x & 9x^2-42xy+49y^2 \\
\qquad\qquad\qquad\quad \underline{+3x} & \underline{\pm 9x^2} \\
\left[\dfrac{-42xy}{6x}=-7y\right] \quad 6x-7y & -42xy+49y^2 \leftarrow \text{remainder} \\
 & \underline{\mp 42xy \pm 49y^2} \\
 & 0 \leftarrow \text{remainder}
\end{array}
$$

$\therefore$ Square root $=\pm(3x-7y)$

Example 30: Find the square root of $4\left(x^2+\dfrac{1}{x^2}\right)+12\left(x+\dfrac{1}{x}\right)+17,\ x\neq 0.$

Solution: $4\left(x^2+\dfrac{1}{x^2}\right)+12\left(x+\dfrac{1}{x}\right)+17=4x^2+\dfrac{4}{x^2}+12x+\dfrac{12}{x}+17$

$=4x^2+12x+17+\dfrac{12}{x}+\dfrac{4}{x^2}$ ← arrange in descending order

$$
\begin{array}{l|l}
 & \quad 2x+3+\dfrac{2}{x} \\ \hline
2x & 4x^2+12x+17+\dfrac{12}{x}+\dfrac{4}{x^2} \\
 & \underline{\pm 4x^2} \\
4x+3 & +12x+17+\dfrac{12}{x}+\dfrac{4}{x^2} \\
 & \underline{\pm 12x \pm 9} \\
4x+6+\dfrac{2}{x} & 8+\dfrac{12}{x}+\dfrac{4}{x^2} \\
 & \underline{\pm 8 \pm \dfrac{12}{x} \pm \dfrac{4}{x^2}} \\
 & 0
\end{array}
$$

$\therefore$ Required square root is $\pm\left(2x+3+\dfrac{2}{x}\right)$

Sometimes the given expression is not a perfect square and to make it a perfect square, we add (subtract) some value to (from) it. Consider the following example.

Example 31: To convert $x^4 + 4x^3 + 10x^2 + 10x + 5$ into a perfect square,

i. What should be added to it?
ii. What should be subtracted from it?
iii. What should be the value of x?

Solution: First we try to find the square root of the given expression as

$$
\begin{array}{r|l}
 & \quad x^2 + 2x + 3 \\
\hline
x^2 & x^4 + 4x^3 + 10x^2 + 10x + 5 \\
 & \pm x^4 \\
\hline
2x^2 + 2x & \quad 4x^3 + 10x^2 + 10x + 5 \\
 & \quad \pm 4x^3 \pm 4x^2 \\
\hline
2x^2 + 4x + 3 & \qquad 6x^2 + 10x + 5 \\
 & \qquad \pm 6x^2 \pm 12x \pm 9 \\
\hline
 & \qquad\qquad -2x - 4
\end{array}
$$

For a perfect square, remainder must be zero, but here remainder is $-2x-4$.
Hence,

i. Expression will be a perfect square if we add $-(-2x-4) = 2x + 4$ in it.
ii. Expression will be a perfect square if we subtract $-2x-4$ from it.
iii. For the value of x put remainder equal to zero as

$$-2x - 4 = 0$$
$$-2x = 4 \quad \text{or} \quad x = -2$$

$\therefore$ For $x = -2$, expression will be a perfect square.

Example 32: For what values of m and n, $9x^4 - 24x^3 - 14x^2 + mx + n$ is a complete square or perfect squre?

Solution: First we try to find the square root of $9x^4 - 24x^3 - 14x^2 + mx + n$.

$$\begin{array}{r|l}
 & 3x^2 - 4x - 5 \\
\hline
3x^2 & 9x^4 - 24x^3 - 14x^2 + mx + n \\
 & \pm 9x^4 \\
\hline
6x^2 - 4x & -24x^3 - 14x^2 + mx + n \\
 & \mp 24x^3 \pm 16x^2 \\
\hline
6x^2 - 8x - 5 & -30x^2 + mx + n \\
 & \mp 30x^2 \pm 40x \pm 25 \\
\hline
 & mx - 40x + n - 25 \\
 & = x(m - 40) + (n - 25)
\end{array}$$

The, given expression will be a complete square if the remainder is zero. This is only possible if,

$m - 40 = 0$ and $n - 25 = 0$

or $m = 40$ and $n = 25$

Hence, for $m = 40$ and $n = 25$ given expression will be a complete square

EXERCISE 4.7

Find the square root of the following by factorization.

1. $16y^2 - 56y + 49$
2. $25a^4 - 30a^3 + 9a^2$
3. $\left(x^2 - \dfrac{1}{x^2}\right)^2 + 4\left(x^2 - \dfrac{1}{x^2}\right) + 4,\ x \neq 0$
4. $\left(a^2 + \dfrac{1}{a^2}\right) - 8\left(a - \dfrac{1}{a}\right) + 14,\ a \neq 0$
5. $(a + 2)(a + 4)(a + 6)(a + 8) + 16$

Find the square root of the following by division.

6. $x^4 + 8x^3 + 20x^2 + 16x + 4$
7. $x^4 + 10x^3 + 31x^2 + 30x + 9$
8. $49b^4 + 18a^2b^2 + 4a^3b + a^4 + 28ab^3$
9. $4x^4 - 12x^3 + 29x^2 - 30x + 25$
10. $1 - 10x + 27x^2 - 10x^3 + x^4$
11. $x^4 + \dfrac{1}{x^4} + 4x^2 - \dfrac{4}{x^2} + 2,\ x \neq 0$
12. $a^2 - 8a + 2 + \dfrac{56}{a} + \dfrac{49}{a^2},\ a \neq 0$
13. $x^4 - 2x^3 + \dfrac{3x^2}{2} - \dfrac{x}{2} + \dfrac{1}{16}$
14. $4x^4 + 32x^2 + 96 + \dfrac{64}{x^4} + \dfrac{128}{x^2}$

15. To make $a^4 - 10a^3 + 27a^2 - 9a + 2$ a perfect square:

i. What should be added in it? **ii.** What should be subtracted from it?

iii. What will be the value of a?

16. Find the values of p and q if $x^4 - 12x^3 + px + q$ is a complete square.

17. For what value of k, the expression $y^4 + 4y^2 + k + \frac{8}{y^2} + \frac{4}{y^4}$ becomes a perfect square, where $y \neq 0$.

4.5.3 Application of Factorization in Daily Life

We use various basic principles of mathematics quite unknowingly in our daily life. Like, we are always using addition, subtraction, division and multiplication everywhere, from restaurants to public transport. When children learn about numbers and basic mathematics. This happens because, with time, we become familiar with the concepts. They can frequently apply these concepts in solving their real life problems. Factorization is a similar example of this, we have a bunch of real-life examples where we use factorization extensively, making our daily lives easier.

Example 33:

Mustafa is working on a space project to make a cuboid with the volume as the difference of 2 cubes of sides x and y respectively. Help him to find:

a. Expression for volume of the required cuboid.

b. Expression for any one side of required cuboid.

c. Expression for Area of any one surface of required cuboid.

Solution:

Mustafa is working with cubes of volumes:

Volume 1: x^3 Volume 2 : y^3

Difference of 2 volumes $= x^3 - y^3$

a. Volume of required cuboid $= x^3 - y^3$

b. $x^3 - y^3 = (x-y)(x^2+xy+y^2)$ (by factorizing)

The volume of a cuboid is factorized in one linear and one quadratic factor. So the expression for length is "$x-y$"

c. The quadratic factor in above factorization represents Area of a surface. i-e Area of one surface is $x^2 + xy + y^2$.

Example 34:

Volume of a cubical container is given by expression $(x^3 - 6x^2y + 12xy^2 - 8y^3)\ m^3$. Find:

a. Expression for length of each side

b. Expression for area of the base

c. Expression for total surface area

d. Expression for cost of painting all outer surfaces at the rate of Rs.50/m^2.

Solution:

Volume of cube = $l^3 = x^3 - 6x^2y + 12xy^2 - 8y^3$

$= (x - 2y)^3$

a. Length of each side is $(x - 2y)$ m

b. Base area = $l^2 = (x - 2y)^2\ m^2$

c. Surface area = $6l^2 = 6\,(x - 2y)^2\ m^2$

d. Cost = rate × surface area = Rs. $50 \times 6 \times (x - 2y)^2$

= Rs. $300\,(x - 2y)^2$

EXERCISE 4.8

1. In a map of an industry, expression of area of a rectangular veranda is given by $(x^2 - 2x - 3)\text{m}^2$. Find:
   a. expressions for both dimensions of veranda.
   b. expression for perimeter of veranda.
   c. expression for cost of fencing veranda @ Rs. 200/m.
   d. expression for the cost of carpeting veranda floor @ Rs. 250/m².

2. Area of a square shaped surface of a machine is given by the expression $(25x^2 - 30x + 9)\text{m}^2$. Find:
   a. expression for the length of the surface.
   b. expression for the boundary of the surface.
   c. expression for the cost of polishing the surface of machine @ Rs75/m².
   d. expression for the cost of edging around 2 sides of the machine surface @ Rs 28/m.

3. Volume of a cubical oil tank in an oil refinery is expressed as $(125x^3 - 150x^2 + 60x - 8)\ \text{m}^3$. Find:
   a. expression for height of oil tank.
   b. expression for surface area of oil tank.
   c. expression for painting it from outside @ Rs32/m².
4. A mechanical engineer working on wheels of a machine, finds their areas given by $A_1 = \pi x^2 - 6\pi x + 9\pi$ and $A_2 = \pi x^2 - 10\pi x + 25\pi$. Help him find radii of both wheels.
5. A machine has two squared shape pressers with areas expressed by $25m^2$ and $36n^2$ respectively. Difference of these areas describes a rectangular presser. Find dimension of the rectangular pressers.
6. Distance covered by a missile to hit the target is given by expression $(x^2 + 5x + 6)$m. Find:
   a. the possible expression for speed of missile
   b. the possible expression for time to reach the target

KEY POINTS

- Factorization of expressions of the following types.

  Type-I: $ka + kb + kc = k(a + b + c)$

  Type-II: $ac + ad + bc + bd = (a + b)(c + d)$

  Type-III: $a^2 + 2ab + b^2 = (a + b)^2$ and $a^2 - 2ab + b^2 = (a - b)^2$

  Type-IV: $a^2 - b^2 = (a + b)(a - b)$

  Type-V: $a^2 + 2ab + b^2 - c^2 = (a + b + c)(a + b - c)$ and

  $a^2 - 2ab + b^2 - c^2 = (a - b + c)(a - b - c)$

  Type-VI: $a^4 + a^2b^2 + b^4 = (a^2 + b^2 + ab)(a^2 + b^2 - ab)$

  $a^4 + 4b^4 = (a^2 + 2b^2 + 2ab)(a^2 + 2b^2 - 2ab)$

  Type-VII: $x^2 + px + q$ (factorize it write p as the sum of the factors of q)

  Type-VIII: $ax^2 + bx + c$ (factorize it write b as the sum of the factors of ac)

  Type-IX: $\begin{cases} (ax^2 + bx + c)(ax^2 + bx + d) + k \\ (x + a)(x + b)(x + c)(x + d) + k \quad \text{(where } a + b = c + d) \\ (x + a)(x + b)(x + c)(x + d) + kx^2 \end{cases}$

  Type-X: $a^3 + 3a^2b + 3ab^2 + b^3 = (a + b)^3$ and $a^3 - 3a^2b + 3ab^2 - b^3 = (a - b)^3$

  Type-XI: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ and $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

- HCF of two or more polynomials is a highest degree polynomial which divides the given polynomials exactly.
- LCM of two or more polynomials is a least degree polynomial which is exactly divisible by the given polynomials.
- Product of two polynomials P and Q = Product of their HCF and LCM i.e.

  $P \times Q = HCF \times LCM$
- An algebraic expression of the form $\frac{P}{Q}$ where P and Q are two expressions and $Q \neq 0$ is called an algebraic fraction.
- A fraction having rational expression in its denominator or numerator or both, is called a complex fraction.
- In case of division, after converting the fractions into multiplication form, follow the same rules for simplification as applied in the product of algebraic fractions.
- Use 'DMAS' rule while simplifying the algebraic fractions having more than one operation.
- Square root of an algebraic expression is defined as one of its equal factors.
- Square root of an algebraic expression 'P' is another expression 'Q' which, when squared, gives 'P'

  i.e., if $P = (\pm Q)^2$ then, $+Q$ and $-Q$ both are square roots of P.

MISCELLANEOUS EXERCISE 4

1. Encircle the correct option in the following.

(i) Factors of $-2-a+a^2$ are

(a) $(a-2)(a-1)$ (b) $(a+1)(a+2)$ (c) $(a+2)(a-1)$ (d) $(a+1)(a-2)$

(ii) Factorization of $x^2-x+\frac{1}{4}$ is

(a) $\left(x+\frac{1}{2}\right)\left(x-\frac{1}{2}\right)$ (b) $\left(x+\frac{1}{2}\right)(x-1)$

(c) $\left(x-\frac{1}{2}\right)(x+1)$ (d) $\left(x-\frac{1}{2}\right)\left(x-\frac{1}{2}\right)$

(iii). $(x^2+m^2)(x+m)(x^4+m^4)(x-m)$ is the factored form of

(a) x^4-m^4 (b) x^8+m^8 (c) x^8-m^8 (d) x^4+m^4

(iv). a^4+64b^4 is the product of

(a) $a^2-4ab+8b^2$ and $a^2+4ab+8b^2$ (b) a^2+8b^2+4ab and a^2-8b^2+4ab

(c) $(a^2+8b^2)^2$ (d) none of these

(v). $x(2a+3b+7)+x(2a+3b+5)$ equals

(a) $2x(2a+3b+6)$ (b) $4x(2a+3b+6)$

(c) $x(2a+3b+7)(2a+3b+5)$ (d) $(x+x)(2a+3b+12)$

(vi) Which is the highest common factor of $-12x^2y^2$, $6xy^3$, $24x^2y^2$?

(a) $-6xy$ (b) $6x^2y^3$ (c) $6xy^2$ (d) $-24x^2y^3$

(vii) What is the least common multiple of $12x^2y^2$, $6xy^3$, $24x^2y^2$?

(a) $6xy$ (b) $6x^2y^3$ (c) $6xy^2$ (d) $24y^3x^2$

(viii) What is the highest common factor of $7x-6xy$ and $5xy^3-3x^2$?

(a) $(7-6y)(5y^3-3x)$ (b) $(7x-6xy)(5y^3x-3x^2)$

(c) x (d) $x(7-6y)(5y^3-3x)$

(ix) Least common multiple of $7x-6xy$ and $5y^3x-3x^2$ is:

(a) $(7-6y)(5y^3-3x^2)$ (b) $(7x-6xy)(5y^3-3x^2)$

(c) x (d) $x(7-6y)(5y^3-3x)$

(x) HCF of $7x^3-8y^3$ and $3x^3-5y^3$ is:

(a) 1 (b) $(7x^3-8y^3)(3x^3-5y^3)$

(c) $7x^3-8y^3$ (d) $3x^3-5y^3$

(xi) LCM of $7x^3 - 8y^3$ and $3x^3 - 5y^3$ is:

(a) 1 (b) $(7x^3 - 8y^3)(3x^3 - 5y^3)$

(c) $7x^3 - 8y^3$ (d) $3x^3 - 5y^3$

(xii) What is the product of $\frac{uv^2}{3w^3}$ and $\frac{6w^4}{u^2v^3}$?

(a) $\frac{uv^2}{3w^2} \times \frac{u^2v^3}{6w^4}$ (b) $\frac{2w}{uv}$ (c) $\frac{uv}{2w}$ (d) both a & c

(xiii) What is the quotient of $\frac{3y^2}{10} \div \frac{y^3}{2}$?

(a) $\frac{3y^5}{20}$ (b) $\frac{3}{5y}$ (c) $\frac{3y^2}{10} \div \frac{2}{y^3}$ (d) $\frac{5y}{3}$

(xiv) What is the sum of $\frac{2a}{a^2-1}$ and $\frac{-a}{a^2-1}$?

(a) $\frac{3a}{a^2-1}$ (b) $\frac{a}{a^2-1}$ (c) $\frac{2a-a}{(a^2-1)+(a^2-1)}$ (d) $\frac{-2a}{a^2-1}$

(xv) What is the difference of $\frac{-3x}{x+y}$ and $\frac{x}{x+y}$?

(a) $\frac{-2x}{x+y}$ (b) $\frac{-3x-x}{2x+2y}$ (c) $\frac{3x^2}{x+y}$ (d) $\frac{-4x}{x+y}$

(xvi) If product of two polynomials is $(a-b)^2(a^2+ab+b^2)$ and their HCF is $a-b$, what is their LCM?

(a) $a^3 - b^3$ (b) $(a-b)^3(a^2+ab+b^2)$

(c) $(a-b)^2(a^2+ab+b^2)$ (d) a^2+ab+b^2

(xvii) If product of HCF and LCM of two polynomials is $(x^3 - y^3)(x+y)$ then what will be the product of these polynomials?

(a) $x^4 - y^4$ (b) $(x-y)(x^2+y^2)$

(c) $(x^2-y^2)(x^2+xy+y^2)$ (d) $(x^2-y^2)(x^2-xy+y^2)$

(xviii) What is the square root of $36x^6y^{16}$?

(a) $6x^6y^{16}$ (b) $6xy$ (c) $6x^3y^8$ (d) $16x^3y^8$

(xix) What is the square root of $(15x^2 - 7y^2)^4$?

(a) $\pm(15x^2 - 7y^2)^2$ (b) $\pm(15x^2 - 7y^2)$

(c) $\pm(15x - 7y)^2$ (d) $\pm(15x - 7y)$

(xx) What is the square root of $\left[-\left(2x+\frac{1}{x}+1\right)\right]^2$?

(a) $\pm\left(2x+\frac{1}{x}+1\right)^2$ (b) $\left(2x+\frac{1}{x}+1\right)^2$ (c) $2x+\frac{1}{x}+1$ (d) $\sqrt{2x+\frac{1}{x}+1}$

Factorize the following.

2. $(a^2-5)^2-13(a^2-5)+36$
3. $6x^2+19x+15$
4. $(x+1)(x-3)(x-5)(x-9)+44$
5. x^4-12x^2+16
6. $(3x^2+4x-5)(3x^2-2+4x)-4$
7. $2m^4+m^2n^2-3n^4$
8. $x^4+10x^2y^2-56y^4$
9. $x^2+2ax-bx-2ab$
10. $a^{12}-b^{12}$
11. $28x^4y+64x^3y-60x^2y$
12. $4(x-y)^3-(x-y)$
13. $x^3p^2-8y^3p^2-4x^3q^2+32y^3q^2$
14. $x^4+y^4-7x^2y^2$
15. Find the highest common factor of the following.
    (i) $x^3+3x^2-8x-24$, x^3+3x^2-3x-9
    (ii) $3x^4-3x^3-2x^2-x-1$, $9x^4-3x^3-x-1$
16. Find the LCM of the following.
    (i) $2x^2+3x+1$, $2x^2+5x+2$, x^2+3x+2
    (ii) $3x^2+11x+6$, $3x^2+8x+4$, x^2+5x+6
17. Find the HCF and LCM of the following expressions.
    $a(a+c)-b(b+c)$, $b(b+a)-c(c+a)$, $c(c+b)-a(a+b)$
18. Find the square root of $\frac{9a^2}{x^2}-\frac{6a}{5x}+\frac{101}{25}-\frac{4x}{15a}+\frac{4x^2}{9a^2}$
19. Simplify the following (all the expressions in the denominator are non-zero).
    (i) $\frac{x^2+x-2}{x^2-x-20}\times\frac{x^2+5x+4}{x^2-x}\div\left(\frac{x^2+3x+2}{x^2-2x-15}\times\frac{x+3}{x^2}\right)$
    (ii) $\frac{1}{x+1}-\frac{1}{(x+1)(x+2)}+\frac{1}{(x+1)(x+2)(x+3)}$
20. Find the values of a and b if
    $x^4+ax^3+bx^2-4x+4$ is a perfect square.

UNIT
05

LINEAR EQUATIONS AND INEQUALITIES

In this unit the students will be able to:

- Recall linear equation in one variable.
- Solve linear equation with rational coefficients.
- Reduce equations, involving radicals, to simple linear form and find their solutions.
- Define absolute value.
- Solve the equation, involving absolute value, in one variable.
- Define inequalities (>, <) and (≥, ≤).
- Recognize properties of inequalities (i.e. trichotomy, transitive, additive and multiplicative).
- Solve linear inequalities with rational coefficients.

We can use equations and formulae to model a variety of real life problems and situations. Business, industry, science, sports, travel, architecture and banking are some of the areas that really depend upon equations and inequalities to find the solutions to their problems. For example, consider the following problem taken from our daily life. Sarim and Raahim are football players in their school team.

If we want to compare their scores for the season, only one of the following statements will be true.

- Sarim scored less umber of goals than Raahim.
- Sarim scored the same number of goals as Raahim.
- Sarim scored more number of goals than Raahim.

Let x and y represent the number of goals scored by Sarim and Raahim respectively. We can compare their scores by using an inequality or an equation as given below.

$x < y$ or $x = y$ or $x > y$.

5.1 Linear Equations in One Variable

A **linear equation in one variable** is an equation that can be written in the standard form as **$ax + b = 0$** where a, b are real numbers and $a \neq 0$ e.g.
$2x + 3 = 0$, $5x + 3 = 5$

Key Fact

1. Exponent of the variable in linear equation is always 1. The equation $x^2 = 4$ is non-linear because exponent of the variable is not '1'.
2. Linear equations do not involve product of the variables. The equation $xy = 14$ is a non-linear equation.
3. Linear equation is also called **equation of degree one**.

5.1.1 To Find the Solution of Linear Equations

The **solution** of a linear equation in one variable is a replacement for the variable that makes the equation true.

A linear equation in one variable (in standard form) has exactly one solution. For the solution of such equations, we have to isolate the variable on either side of equal sign by a sequence of **equivalent equations.**

Key Fact

Two or more equations, which have the same solutions, are called equivalent equations. e.g. $2x + 3 = 4$ and $2x = 1$ are two **equivalent equations**.

We follow properties of equality i.e. addition, subtraction, multiplication and division properties while solving first degree linear equations.

Example 1: Solve the following equation for x.

$$-7x + 24 = 3$$

Solution:

$-7x + 24 = 3$ ⟵ original equation

$-7x + 24 - 24 = 3 - 24$ ⟵ subtract 24 from both sides

$-7x = -21$ ⟵ divide both sides by -7

$x = 3$

Check: To check this root, we replace the variable x by its value in the original equation and simplify both sides.

$-7x + 24 = 3$ ⟵ original equation

$-7(3) + 24 \overset{?}{=} 3$ ⟵ replace 'x' by 3

$-21 + 24 \overset{?}{=} 3$ ⟵ simplify L.H.S

$3 = 3$ ⟵ solution is checked

Thus, $x = 3$ is required root.

Point to Ponder!

While checking any solution, it is better to write a question mark over the equal sign just to indicate that we are not sure of the validity of the equation.

5.1.2 Solving Linear Equations Involving Fractions

The following examples illustrate this method.

Example 2: Solve $\frac{3x}{5}-\frac{1}{2}=\frac{x}{4}+1$

Solution: $\frac{3x}{5}-\frac{1}{2}=\frac{x}{4}+1$

LCM of 5, 2 and 4 = 20.

$$20\times\frac{3x}{5}-20\times\frac{1}{2} = 20\times\frac{x}{4}+20$$
$$12x-10 = 5x+20$$
$$12x-5x = 20+10$$
$$7x = 30$$
$$x = \frac{30}{7}$$

Thus, $x=\frac{30}{7}$ is the required root of the given equation.

Historya Mystery

Finding solution of equations has been a principal aim of mathematics for thousands of years. However, the equal sign did not occur in any text until 1557.

Example 3: Solve $\frac{2x+3}{5}=\frac{3-4x}{8}$

Solution: $\frac{2x+3}{5}=\frac{3-4x}{8}$

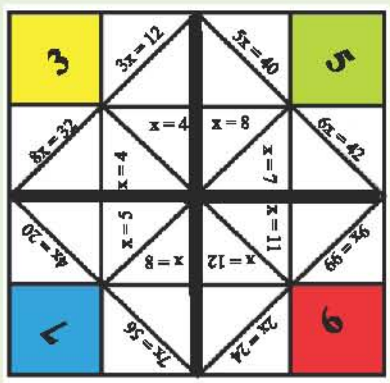
By cross multiplication we get:

$$8(2x+3) = 5(3-4x)$$
$$16x+24 = 15-20x$$
$$16x+20x = 15-24$$
$$36x = -9$$
$$x = -\frac{9}{36}=-\frac{1}{4}$$

Thus, $x=-\frac{1}{4}$ is the root of given equation.

Funmatics

Help students to make such a cootie catcher to play with friends for reinforcement of solution of linear equations.



Example 4: Solve $0.7(x-1)-0.5x=1.1$

Solution:
$$0.7(x-1)-0.5x = 1.1$$
$$0.7x-0.7-0.5x = 1.1$$
$$0.2x-0.7 = 1.1$$
$$0.2x = 1.8$$
$$2x = 18$$
$$x = 9$$

Check: Replace x by its value in the original equation.

$$0.7(9-1)-0.5(9) \stackrel{?}{=} 1.1$$
$$5.6-4.5 \stackrel{?}{=} 1.1$$
$$1.1 = 1.1$$

Thus, $x = 9$ is the required solution of the given equation.

EXERCISE 5.1

Solve the following linear equations in one variable.

1. $5x - 2 - x = 4 - 3x - 27$
2. $4a - 3(5a - 14) = 5(7 + a) - 9$
3. $7(2 - 5x) + 27 = 18x - 3(8 - 4x)$
4. $\frac{5x}{4} + \frac{1}{2} = 0$
5. $\frac{x-2}{2} + \frac{x+10}{9} = 5$
6. $\frac{4(x+2)}{3} - \frac{6(x-7)}{7} = 12$
7. $\frac{x}{2} + \frac{x}{3} - \frac{x}{4} + \frac{x}{5} = 7\frac{5}{6}$
8. $\frac{y+1}{3} + \frac{y+1}{2} = 2 - \frac{y+3}{2}$
9. $\frac{1}{5}(x-8) + \frac{4+x}{7} = 7 - \frac{23-x}{5}$
10. $\frac{1}{2y} - \frac{1}{6} = \frac{1}{4y} - 1 - \frac{1}{y}$; $y \neq 0$
11. $4 - 0.3(1 - x) = 7$
12. $0.5x = 6.3 - 0.2x$
13. $1.3x - 0.2 = 0.3x - 1.5$

5.2 Linear Equations Involving Radicals

5.2.1 Radical Equations

An equation in which the unknown letter (variable) appears under a radical sign is called a **radical equation.**

Examples: $\sqrt{x+1} = 7$, $\sqrt{x} = 9$, $\sqrt{x+2} = 5$, $\sqrt{2x-3} = \sqrt{x+5}$

Key Fact

- If two numbers are equal, then their squares are also equal i.e. if $x = y$ then $x^2 = y^2$
- If the squares of two numbers are equal, the numbers may or may not be equal. e.g. $(-5)^2 = (5)^2$ but $-5 \neq 5$

5.2.2 To Solve a Radical Equation

I. Arrange the terms such that a term with a radical sign is by itself on one side of the equation.

II. Square both sides of the equation.

III. Solve the resulting linear equation for corresponding variable.

IV. Check the solution in the original equation.

Key Fact

The equation in which after isolating the radical term, if radical term is equal to a negative number, such equation has no solution in real numbers.

Example 5: Solve $\sqrt{x}+3=7$

Solution:
$$\sqrt{x}+3=7$$
$$\sqrt{x}=4$$
$$(\sqrt{x})^2=(4)^2$$
$$x=16$$

Check: Replace x by 16 in the original equation.
$$\sqrt{16}+3\overset{?}{=}7$$
$$4+3=7$$
$$7=7 \longleftarrow \text{solution is checked}$$

Hence, $x=16$ is required solution.

While squaring both sides of a radical equation it is possible to get an extra root called extraneous root that does not satisfy the original equation. Therefore, it is necessary to check every root by substituting it into the original equation. Consider the following example for such case.

Example 6: Solve $4+2\sqrt{3y+1}=3$

Solution:
$$4+2\sqrt{3y+1}=3$$
$$2\sqrt{3y+1}=-1 \;\ldots\ldots\text{(i)}$$
$$(2\sqrt{3y+1})^2=(-1)^2$$
$$4(3y+1)=1$$
$$12y+4=1$$
$$12y=-3$$
$$y=-\frac{1}{4}$$

Check:
$$4+2\sqrt{3(-\frac{1}{4})+1}\overset{?}{=}3$$
$$4+2\sqrt{\frac{-3}{4}+1}\overset{?}{=}3$$
$$4+2\sqrt{\frac{1}{4}}\overset{?}{=}3$$
$$4+1\overset{?}{=}3$$
$$5\neq 3$$

Thus, $y=-\frac{1}{4}$ is an extraneous root and solution set is ϕ.

Note: In example 6, there is no need to solve the equation after step (i). We can directly say that equation has no solution.

EXERCISE 5.2

Reduce the following radical equations into simple linear equations then find their solution. In case of extraneous solution, write ϕ for the solution set.

1. $\sqrt{2x}=4$
2. $\sqrt{x-3}=2$
3. $\sqrt{x-5}=3$
4. $\sqrt{2x+1}=9$
5. $\sqrt{5x-4}=14$
6. $\sqrt{3x-5}=-10$
7. $\sqrt{y+4}-3=2$
8. $5-\sqrt{2x-1}=0$

9. $\sqrt{y+1} - 12 = -10$
10. $\sqrt{5t-2} = \sqrt{3t+4}$
11. $\sqrt{9-2x} = \sqrt{5x-12}$
12. $12 - \sqrt{y+1} = 14$
13. $4\sqrt{z} + 8 = 40$
14. $\sqrt{\frac{a+6}{a+2}} = \sqrt{\frac{a+2}{a-1}}$
15. $\sqrt{\frac{z}{z+3}} = \sqrt{\frac{z+2}{z+6}}$
16. $\sqrt{5x-4} = \sqrt{7x+2}$

5.3 Equations Involving Absolute Value

5.3.1 Defining an Absolute Value

The **absolute value** of a number is its distance from 0 on the number line. If x is any point on the number line then its distance from 0 is denoted by $|x|$. The two vertical bars are called **absolute value bars.** Since distance between any two points is always a positive number or zero, thus the absolute value of a number is always a positive number or zero e.g. distance from 0 to 5 or from 0 to – 5 is 5 units on the number line.

Thus, $|5| = 5$ and $|-5| = 5$

The absolute value of a real number x, written as $|x|$, is defined as

- $|x| = x$, if $x \geq 0$
- $|x| = -x$ if $x < 0$ e.g. $|9| = 9$ or $|-3| = -(-3) = 3$

5.3.2 Solution of Absolute Value Equations

An equation that contains a variable inside the absolute value bars is called an **absolute value equation.**

e.g. $|x| + 1 = 5$, $|x-3| = 4$

To solve the equations involving absolute value, we apply the basic definition of absolute value.

Example 7: Solve the equation $|x| = 8$.

Solution: To solve such equation we have to consider both the possible values of the number with absolute value.

Thus, if $|x| = 8$ then $x = 8$ or $x = -8$.

The solution set is $\{-8, 8\}$.

Example 8: Solve the equation $|x| = -6$.

Solution There is no real number x such that $|x| = -6$. So, this equation has no solution. Hence the solution set is ϕ.

Conclusions

An absolute value equation of the form $|ax+b| = c$, where a, b and c are real numbers, $a \neq 0$ and $c > 0$ is equivalent to two equations:

$$ax + b = c \quad \text{or} \quad ax + b = -c.$$

For the required solution set of the given absolute value equation, we solve both the equations separately.

Example 9: Solve the equation:

$$|2x+5| = 11, \text{ where } x \in R.$$

Solution:

Step-I: Remove the absolute value bars and write two equations as

$$2x + 5 = 11 \quad \text{or} \quad 2x + 5 = -11$$

Step-II: Now solve both the equations for x.

$2x + 5 = 11$	$2x + 5 = -11$
$2x = 11 - 5$	$2x = -11 - 5$
$2x = 6$	$2x = -16$
$x = 3$	$x = -8$

Step-III: Hence 3 and –8 are roots of the absolute value equations. Thus, the solution set is $\{3, -8\}$

- The equation $|x| = k$ and $k > 0$ has two solutions k and $-k$.
- The equation $|x| = 0$ has one solution, namely $x = 0$.
- The equation $|x| = k$ and $k < 0$ has no solution and the solution set in this case is ϕ.

Example 10: Solve the absolute value equation.

$$\frac{|10-x|}{5} = \frac{|2x-5|}{2}, \quad \text{where } x \in R$$

Solution: Before we remove the absolute value bars, try to isolate the absolute value expressions on either side as

$$\frac{|10-x|}{5} = \frac{|2x-5|}{2}$$

$$\frac{|10-x|}{|2x-5|} = \frac{5}{2}$$

$$\left|\frac{10-x}{2x-5}\right| = \frac{5}{2} \longleftarrow \because \frac{|a|}{|b|} = \left|\frac{a}{b}\right|$$

$\frac{10-x}{2x-5} = \frac{5}{2}$	or	$\frac{-10x}{2x-5} = -\frac{5}{2}$
$\frac{10-x}{2x-5} = \frac{5}{2}$		$\frac{10-x}{2x-5} = -\frac{5}{2}$
$2(10-x) = 5(2x-5)$		$2(10-x) = -5(2x-5)$

Key Fact

- $|ab| = |a|\,|b|$
- $\frac{|a|}{|b|} = \left|\frac{a}{b}\right|$
- $|a| + |a| = 2\,|a|$

$$\begin{aligned} 20-2x &= 10x-25 \\ -12x &= -45 \\ x &= \frac{45}{12}=\frac{15}{4} \end{aligned} \qquad \begin{aligned} 20-2x &= -10x+25 \\ 8x &= 5 \\ x &= \frac{5}{8} \end{aligned}$$

Thus, the solution set is $\left\{\frac{15}{4}, \frac{5}{8}\right\}$.

Example 11: Solve the following absolute value equaiton.

$$|a-1| = |2a-3|, \quad a \in R$$

Solution:

By removing the absolute value bars we get two equations as:

$$\begin{aligned} a-1 &= 2a-3 \\ -a &= -2 \\ a &= 2 \end{aligned} \quad \text{or} \quad \begin{aligned} a-1 &= -(2a-3) \\ a-1 &= -2a+3 \\ 3a &= 4 \\ a &= \frac{4}{3} \end{aligned}$$

The solution set is $\left\{2, \frac{4}{3}\right\}$.

EXERCISE 5.3

Solve the following absolute value equations, where $x, y, z \in R$.

1. $|x| = \frac{5}{3}$
2. $|x+2| = 6$
3. $|5y-1| = 9$
4. $|x+1| = 2$
5. $|6-3y| = 0$
6. $3|z-2| - 4 = -2$
7. $|2x-1| = 5$
8. $|3x+2| = 7$
9. $\frac{|4x|}{3} = 12$
10. $|5x| + 10 = 5$
11. $\frac{|1-2y|}{4} = 3$
12. $\frac{|x+1|}{2} = \frac{|2x-1|}{3}$
13. $|5x-3| = |x+7|$
14. $|z+3| - 3 = 5 - |z+3|$

5.4 Linear Inequalities (or Inequations) in one Variable

5.4.1 Defining an Inequality

There are many ways in which two expressions may be unequal. The following symbols express some inequalities.

Inequality Symbols

$<$	is less than	e.g. $-5 < 7$
$>$	is greater than	e.g. $10 > -3$
$\leq$	is less than or equal to	e.g. $7 \leq 7$
$\geq$	is greater than or equal to	e.g. $5 \geq 1$
$\neq$	is not equal to	e.g. $3 \neq 5$

History a Mystery

The symbols for "is less than" and "is greater than" were introduced by Thomas Harriot around 1630. Before that $\sqsubset$ and $\sqsupset$ were used for $<$ and $>$ respectively.

A statement that "two algebraic expressions are not equal" is called an "**inequality**" or "**inequation.**"

A **linear inequality in one variable** is an inequality (inequation) that can be written in the standard form of $ax+b<0$ (or $ax+b>0$) where a and b are real numbers and $a \neq 0$.

Examples: $x \leq 3$, $x \geq -2$, $x-5<-10$, $5y-7<3y+9$, $-5>-7$, $-3 \neq x+1$

Remark: Above mentioned definition is also valid for the symbols $\leq$ and $\geq$.
Some examples of linear inequations are $4x+3 \geq 0$, $y>-7$, $8(x-2) \leq 3-5x$ and $x+3<-5$

An inequality written with the symbols $<$ or $>$ is called a **strict inequality.**

5.4.2 Solution of Linear Inequalities (in one variable)

The definitions of **solution** and **solution set** for inequalities are same as for equations.

A **solution** of an inequality is a replacement for the variable that makes the inequality true. **Solution set** of an inequality is the set of all real numbers that satisfy the inequality.

Procedure:
The procedure for solving a linear inequality in one variable is almost identical to that for solving a linear equation. Here, also to isolate the variable, we use "**properties of inequalities**". These properties are similar to the properties of equality but there is one important difference that **when both sides of an inequality are multiplied or divided by a negative number then direction of the inequality symbol is reversed.** e.g.

original inequality	←	$-2<5$
$(-3)(-2)>(-3)(5)$	←	multiplying both sides by -3 and reversing inequality symbol
$6>-15$	←	simplest form

Two or more inequalities that have the same solution set are called **equivalent inequalities** e.g. $x+5<8$ and $x<3$ are two equivalent inequalities.

Example 12: Solve the following inequality.
$8x+9<6x-7$, where $x \in R$.

Solution:
$$8x+9<6x-7$$
$$8x-6x<-7-9$$
$$2x<-16$$
$$x<-8$$

Check Point
In example 13 if $x \in Z$, then what will be the solution set?

The solution contais all real numbers less than – 8.
Check the solution by replacing x with any number less than – 8, for example – 9, as

$$8(-9) + 9 < 6(-9) - 7$$
$$-72 + 9 < -54 - 7$$
$$-63 < -61 \leftarrow \text{true statement}$$

Thus the solution set is $\{x \mid x \in R \wedge x < -8\}$.

Example 13: Solve the inequality.

$$3(-4 + 5y) \leq -8(1 - 2y) + 6, x \in R$$

Solution:
$$3(-4 + 5y) \leq -8(1 - 2y) + 6$$
$$-12 + 15y \leq -8 + 16y + 6$$
$$-12 + 15y \leq 16y - 2$$
$$-y \leq 10$$
$$y \geq -10$$

$\therefore$ Solution Set = $\{x \mid x \in R \wedge y \geq -10\}$

Point to Ponder!

- $5 \geq 1$ is true because $5 > 1$ is true.
- $7 \leq 7$ is true because $7 = 7$ is true.

Example 14: Solve. $\frac{1}{2}x + 3 \geq \frac{1}{4}x + 2, x \in R$

Solution:
$$\frac{1}{2}x + 3 \geq \frac{1}{4}x + 2$$
$$4(\frac{1}{2}x + 3) \geq 4(\frac{1}{4}x + 2) \leftarrow \text{multiply both sides by LCM of denominators}$$
$$2x + 12 \geq x + 8$$
$$x + 12 \geq 8$$
$$x \geq -4$$

The solution set is $\{x \mid x \in R \wedge x \geq -4\}$.

Compound Inequalities

Two inequalities that are joined by the word **"and"** or the word **"or"** are called compound inequalities e.g. $2x < 6$ and $3x + 2 > -4$, $3x + 5 > 7$ or $4x - 1 < 3$

5.4.3 Solution of Compound Inequalities Joined with 'or'

When two inequalities are joined with connective word **'or'**, it is necessary to solve each inequality separately. The solution set of the compound inequality will be the union of both the solution sets i.e., the solution set will satisfy either one or both the inequalities.

Key Fact

A compound inequality containing or is true if at least one of its inequality is true.

Example 15: Solve the following.

$$2x + 3 > 7 \quad \text{or} \quad 4x - 1 < 3; \quad x \in R$$

Solution: We will solve both the inequalities for x separately.

$$2x + 3 > 7 \quad \text{or} \quad 4x - 1 < 3$$
$$2x > 4 \quad \text{or} \quad 4x < 4$$
$$x > 2 \quad \text{or} \quad x < 1$$

$\{x \mid x \in R \wedge x > 2\}$ or $\{x \mid x \in R \wedge x < 1\}$

Now the union of both the solution sets is

$\{x \mid x \in R \wedge x > 2 \text{ or } x < 1\}$

Hence, the solution set of the given compound inequality contains all real numbers greater than 2 or less than 1.

5.4.4 Solution of Compound Inequalities Joined with 'and'

When two inequalities are joined with the connective word **'and'** then the solution set of the compound inequality will be the intersection of both the solution sets i.e. the solution set contains all the solutions that satisfy both of the inequalities.

Example 16: Solve the compound inequality.

$x - 5 \geq -1$ and $x + 3 \leq 10, x \in R$

Solution: Solve both the inequalities for 'x'.

$x - 5 \geq -1$	and	$x + 3 \leq 10$
$x \geq 4$	and	$x \leq 7$
$\{x \mid x \in R \wedge x \geq 4\}$	and	$\{x \mid x \in R \wedge x \leq 7\}$

Intersection of two solution sets is $\{x \mid x \in R \wedge 4 \leq x \leq 7\}$

i.e. Solution set consists of all real numbers that are greater than or equal to 4 and less than or equal to 7.

Key Fact

- Two inequalities $-6 < 5x + 3$ and $5x + 3 < 5$ can be written in combined form as $-6 < 5x + 3 < 5$.
- There is no short way to write a compound inequality containing 'or'.
- Compound inequality containing **and** is true if its both inequalities are true.

Example 17: The sum of two times a number x and 3 is between 5 and 17. Between what two numbers is the given number x?

Solution: Expression 'sum of 2 times a number x and 3' can be written as $2x + 3$. Then given compound inequality is '$5 < 2x + 3 < 17$'.

Now we solve this compound inequality.

$5 < 2x + 3 < 17$

$5 - 3 < 2x < 17 - 3$ ⟵ subtracting 3 from each part

$2 < 2x < 14$

$1 < x < 7$ ⟵ dividing each part by 2

Thus, x lies in between 1 and 7.

Math Play Ground

a. Take students to the play ground.
b. Give each student a paper strip with some equation or inequality written on.
c. Spread solutions of all students in the play ground.
d. Ask them to find the respective answers as a game of treasure hunt.

EXERCISE 5.4

1. (a) Check whether the given value of each variable satisfies the inequality.

i) $5y-12>0;\quad y=3$
ii) $4-2x\leq 9;\quad x=-3$
iii) $5-2x>-4x+5;\quad x=4$
v) $3(z+4)\leq 6;\quad z=-2$
v) $5(x-2)\geq 9x-3(2x-4);\quad x=11$

(b) In the following cases, write each solution in the set notation form.

i) $2\leq x\leq 5$, where $x\in N$
ii) $y<7$, where $y\in N$
iii) $z\leq -3$, where $z\in R$
iv) $x\leq 4$, where $x\in W$
v) $-4<x<\dfrac{-3}{2}$, where $x\in R$

Solve the following inequalities.

2. $3x-2<7,\ x\in N$
3. $6x-5\leq 35-2x,\ x\in W$
4. $16-5y<4(y-1)-7,\ y\in R$
5. $10-(7-y)\geq 3y-9,\ y\in R$

Solve the following compound inequalities, where $x\in R$ (6-10).

6. $5-3x<11$ or $2x+3<-9$
7. $2x+3\leq 9$ and $x-5>-6$
8. $1\leq 7-3x\leq 22$
9. $3x+21<1-x$ or $3x+8>3-2x$
10. $1-5x>16$ and $3-\dfrac{3x}{2}\leq 9$
11. The sum of five times a number x and 10 is less than –35 or greater than –5. What real numbers does x represent?
12. Two times a number decreased by 5 is greater than or equal to the number increased by 8. Find the possible values for the number.

KEY POINTS

- An equation that can be written in the standard form $ax+b=0$ where $a, b\in R$ and $a\neq 0$ is called a linear equation in one variable.
- A solution of a linear equation in one variable is a number replacement for the variable that makes it true.
- Two or more linear equations with the same solutions are called equivalent equations.
- To solve a linear equation with fractions, multiply all the terms by the least common multiple of all denominators to clear the fractions.
- Root of a linear equation that does not satisfy the original equation is called an "extraneous root" of that equation.
- A linear equation in which the variable appears under the radical sign is called a radical equation.
- An equation that contains an absolute value symbol is called an absolute value equation.

- If x is a real number then $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$.
- If two equations have the same absolute value, then they are either equal or opposite.
- A statement that two algebraic expressions are not equal is called an inequality.
- An equality that can be written in the standard form of $ax + b < 0$ where a and b are real numbers and $a \neq 0$ is called a linear inequality.
- A solution set is the set of all solutions of an inequality.
- Two or more inequalities, which have the same solution sets are called equivalent inequalities
- A compound inequality is a relation containing two simple inequalities connected with the words **'and'** or **'or'**.

MISCELLANEOUS EXERCISE 5

1. Encircle the correct option in the following. absolutely correct.

(i). Which one is the standard form of linear equation?
(a) $ax^2 + b = 0$ (b) $ax + b = -c$ (c) $ax + b > 0$ (d) $ax + b = 0$

(ii). The exponent of the variable in linear equation is?
(a) 1 (b) 2 (c) -1 (d) 0

(iii). Which one is the linear equation in one variable?
(a) $ax + y = 0$ (b) $xy + 3 = 0$ (c) $2x + 3 = 0$ (d) $2x^2 + 3 = 0$

(iv). Which one is the solution of $12x + 17 = 65$?
(a) 48 (b) $\frac{82}{12}$ (c) 4 (d) $\frac{65}{12}$

(v). What number must be subtracted from the right side of $7x = 30$ so that 4 is a solution of the resulting equation?
(a) 7 (b) 2 (c) 4 (d) -2

(vi). Which property of equality will be applied to solve the equation $-2x = \frac{2}{5}$?
(a) Addition (b) Subtraction
(c) Division (d) Addition and subtraction

(vii). Which one is the solution set of $5|x| = 25$?
(a) $\{-25, 25\}$ (b) $\{-25\}$ (c) $\{5\}$ (d) $\{-5, 5\}$

(viii). Which is the solution set of $|x| + 7 = 3$?
(a) $\{-4\}$ (b) $\{4, -4\}$ (c) $\{\ \}$ (d) $\{-7, -3\}$

(ix). Which one is the solution set of $\sqrt{5x} = -10$?
(a) $\{\ \}$ (b) $\{-20\}$ (c) $\{20\}$ (d) $\{-2\}$

(x). Which one is the solution set of $\sqrt{3x+1} = 5$?
(a) 25 (b) 8 (c) 24 (d) $\frac{26}{3}$

(xi). Which one is the solution of $\frac{4}{x}-\frac{2}{x}=5$?

(a) -1 (b) $\frac{2}{5}$ (c) $\frac{5}{2}$ (d) zero

(xii). Which one is a strict inequality?

(a) $x+3 \neq 0$ (b) $12x > 5$ (c) $2y-3 \leq 0$ (d) $4x+5 \geq 0$

(xiii). What should be value of 'k' if $x < y$ shows $kx > ky$?

(a) $k=0$ (b) $k>0$ (c) $k<0$ (d) $k \geq 0$

(xiv). Which one is the compound relation?

(a) $1+2x<4+x$ (b) $4x+3>5\frac{3}{5}$ (c) $x+y>5\frac{1}{2}$ (d) $x \leq 0$

(xv). Which one is the solution of $3-\frac{1}{2}x \geq 0$?

(a) $x \geq -6$ (b) $x \leq 6$ (c) $x \geq 6$ (d) $x \leq -6$

(xvi). Which one is the solution of $2(x+6) \leq 3(x+4)$?

(a) $x \leq 0$ (b) $x \geq 0$ (c) $x \leq 24$ (d) $x \geq 24$

Solve the following.

2. $\frac{2x-11}{12}=\frac{2x+10}{12}-\left(\frac{28-2x}{4}-\frac{1}{4}\right)$

3. $\sqrt{a-\frac{1}{2}}=\sqrt{\frac{2a}{5}+\frac{2}{5}}$

4. $\frac{\sqrt{5x-4}-4}{10}=-1$

5. $5-|5y+1|=-9$

6. $\frac{3}{4}x-1 \geq x+1, x \in R$

7. $4(2y+3)-(6y-1)>10,\ y \in R$

8. $\frac{3}{2}x \leq -3$ or $\frac{2}{3}x \geq 4,\ x \in R$

9. The difference between three times a number y and 18 is less than 12 or greater than 39. What real numbers do y represent?

UNIT
06

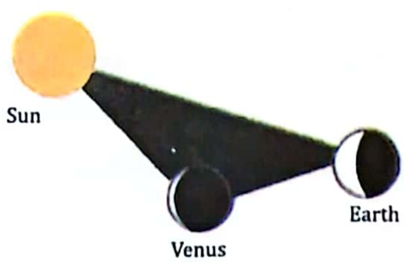
TRIGONOMETRY AND BEARING

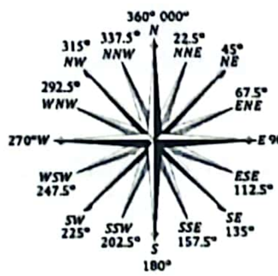
In this unit the students will be able to:

- Measure an angle in sexagesimal and circular systems.
- Convert an angle from D°M'S'' to decimal form and vice versa.
- Convert an angle from sexagesimal to circular system and vice versa.
- Find length of circular arc when radius and central angle are given.
- Find area of the sector of a circle.
- Find the trigonometric ratios of 0°, 30°, 45°, 60°, 90°, 180°, 270° and 360°.
- Prove and apply trigonometric identities.
- Find angles of elevation and depression.
- Solve real life problems related to a right-angled triangle.
- Interpret and use three figure bearings.
- Find bearing of cardinal and intercardinal directions.
- Solve real life problems related to bearing and trigonometry.

Trigonometry is the branch of Mathematics that deals with the relationship of sides with angles in a triangle. With trigonometry, finding out the heights of big mountains or towers is possible. It is also used to find the distance between stars or planets in astronomy and is widely used in physics, architecture, engineering and GPS navigation systems. Trigonometry is based on the principle that **if two triangles have the same set of angles, their corresponding sides are in the same ratio.**

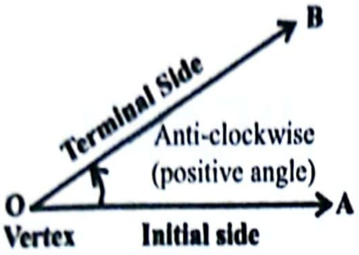
The measurement along with direction of angles plays an important role in satellite navigation. By using trigonometric and bearing principles, navigation has become easier for the military and hikers . Direction compass supports in getting clear picture of the angle between cardinal and intercardinal directions.

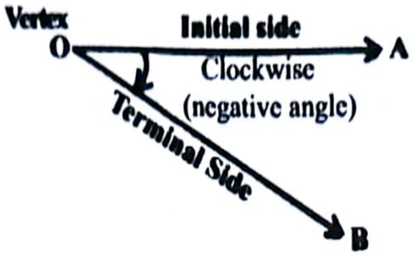




6.1 Measurement of an Angle

Union of two rays having common end point forms an angle. An angle is positive or negative if measured in anti-clockwise or clockwise direction respectively.





Angle in standard position

An angle when drawn in the cartesian plane(rectangular coordinate system) is in **standard** position if its vertex is at the origin and its initial arm is directed along positive **x-axis. Its** terminal arm may lie either on any of the axis or in any of the quadrants.

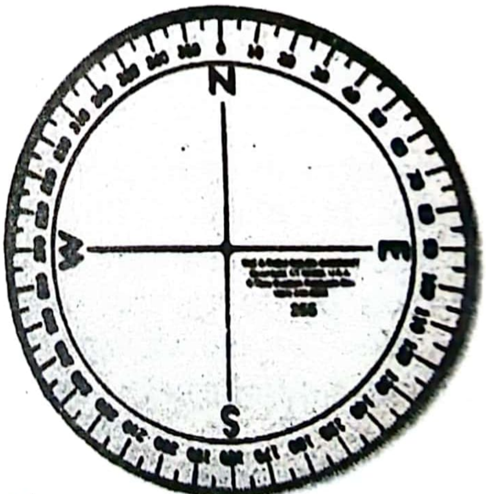
There are three systems of measurement of an angle.

i. Sexagesimal System or English System
ii. Radian measure or Circular System
iii. Centesimal or French System

Centesimal or French system is beyond the scope of this book.

6.1.1 Sexagesimal System

Draw a circle and divide its boundary into 360 equal arcs. The central angle of each arc is of measure 1 degree (1°). To draw an angle of 30°, we need 30 consecutive arcs of 1°. As 90° is one fourth of 360°, so 90 consecutive arcs of 1° form a quadrant of the circle. i.e., central angle of quadrant of a circle is 90°. There are 180 arcs of 1° in a half circle and all 360 arcs form a complete circle. i.e., central angle of half circle is 180° and that of full circle it is 360°. To measure the central angle of an arc whose length is smaller than the length of arc of 1°, we divide the arc of 1° into further 60 equal arcs. The central angle of each of these small arcs is of measure 1 minute denoted as $1'$. So, there are $60'$ in a degree. Similarly, there are 60 seconds($60''$) in a minute.



Thus, $1^\circ = 60'$, $1' = 60''$, $1^\circ = 3600''$.

$$1' = \left(\frac{1}{60}\right)^\circ, 1'' = \left(\frac{1}{60}\right)', 1'' = \left(\frac{1}{3600}\right)^\circ.$$

In sexagesimal system, an angle is measured in Degrees, Minutes and Seconds.

Example 1: Convert the following measures of angles into seconds.

i. $30'$ ii. 60° iii. $45^\circ 45' 45''$

Solution:

i. $30' = 30 \times 60'' = 1800''$

ii. $60^o = 60 \times 60' = 3600' = 3600 \times 60'' = 216000''$

iii. $45^o45'45'' = 45^o + 45' + 45''$

$= 45 \times 3600'' + 45 \times 60'' + 45''$

$= 162000'' + 2700'' + 45'' = 164745''$

Example 2: Convert the following measures of angles into minutes

i. 36^o ii. $45''$ iii. $36^o45'30''$

Solution:

i. $36^o = 36 \times 60' = 2160'$

ii. $45'' = \left(45 \times \frac{1}{60}\right)' = \left(\frac{3}{4}\right)' = 0.75'$

iii. $36^o45'30'' = 36^o + 45' + 30''$

$= 36 \times 60' + 45' + \left(\frac{30}{60}\right)'$

$= 2160' + 45' + 0.5' = 2205.5'$

Example 3: Write the following measures of angles into $D^oM'S''$.

i. 60.5^o ii. 45.125^o iii. $3600''$ iv. $216000''$ v. $20.555'$

Solution:

i. $60.5^o = 60^o + 0.5^o$

$= 60^o + 0.5 \times 60'$

$= 60^o + 30' = 60^o30'$

ii. $45.125^o = 45^o + 0.125^o$

$= 45^o + 0.125 \times 60'$

$= 45^o + 7.5'$

$= 45^o + 7' + 0.5 \times 60'' = 45^o7'30''$

iii. $3600'' = \left(\frac{3600}{60}\right)' = 60' = 1^o = 1^o0'0''$

iv. $216000'' = \left(\frac{216000}{60}\right)' = 3600' = \left(\frac{3600}{60}\right)^o = 60^o0'0''$

v. $20.555' = 20' + 0.555'$

$= 20' + 0.555 \times 60''$

$= 20' + 33.3'' = 0^o20'33''$

Example 4: Write the following measures of angles in degrees correct to 4 decimal places.

i. $60^o30'30''$ ii. $75^o25'35''$

Solution:

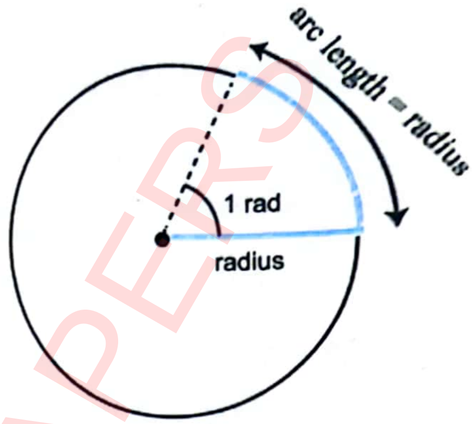
i. $60^o30'30'' = 60^0 + \left(\frac{30}{60}\right)^0 + \left(\frac{30}{3600}\right)^0$

$= 60^0 + 0.5^0 + 0.0083^0 = 60.5083^0$

ii. $75^o25'35'' = 75^0 + \left(\frac{25}{60}\right)^0 + \left(\frac{35}{3600}\right)^0$

$= 75^0 + 0.4167^0 + 0.0097^0 = 75.4264^0$

6.1.2 Circular System or Radian Measure



The ratio between length of circular arc and radius is called radian. If length of an arc is equal to radius of circle, then their ratio is 1 radian. i.e., one radian is measure of central angle of an arc whose length is equal to radius of the circle.

The ratio between circumference and radius of a circle is 2π, therefore measure of a complete angle in circular system is 2π radians.

In circular system, an angle is measured in radians.

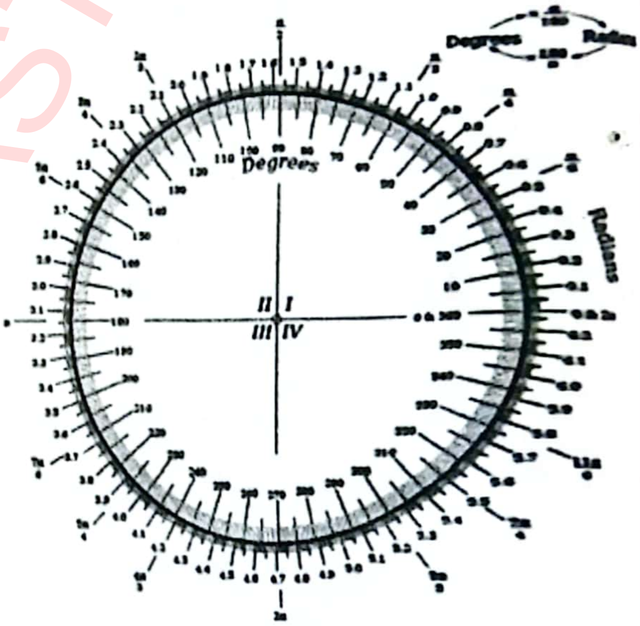
In sexagesimal system measure of a complete angle is 360^0,

so $\quad 2\pi$ radians $= 360^0$

π radians $= 180^0$

$$1 \text{ radian} = \left(\frac{180}{\pi}\right)^0$$

$$\approx 57^0 17' 45''$$



Similarly,

$$1^0 = \frac{\pi}{180} \text{ radians} \approx 0.0175 \text{ radians}$$

To convert measure of an angle given in radians to degrees, multiply its value by $\left(\frac{180}{\pi}\right)^0$ and to convert measure of an angle given in degrees to radians, multiply its value by $\frac{\pi}{180}$ radians.

Example 5:

Convert the following measures of angles in sexagesimal system ($D^0 M' S''$).

i. 2.125 radians ii. 0.5 radians iii. $\frac{\pi}{3}$ radians iv. $\frac{3\pi}{4}$ radians

Solution:

i. $\quad 2.125 \text{ radians} = 2.125 \times \left(\frac{180}{\pi}\right)^0$

$\approx 2.125 \times 57.2958^0$

$\approx 121.7535^0$

$= 121^0 + 0.7535 \times 60'$

$= 121^0 + 45.21'$

$= 121^0 + 45' + 0.21 \times 60'' = 121^0 45' 13''$

ii. $0.5 \text{ radians} = 0.5 \times \left(\frac{180}{\pi}\right)^0$

$\approx 0.5 \times 57.2958^0$

$\approx 28.6479^0$

$= 28^0 + 0.6479 \times 60'$

$= 28^0 + 38.874'$

$= 28^0 + 38' + 0.874 \times 60'' = 28^0 38' 52''$

iii. $\frac{\pi}{3} \text{ radians} = \frac{\pi}{3} \times \left(\frac{180}{\pi}\right)^0$

$= \left(\frac{180}{3}\right)^0 = 60^0$

iv. $\frac{3\pi}{4} \text{ radians} = \frac{3\pi}{4} \times \left(\frac{180}{\pi}\right)^0$

$= \left(\frac{540}{4}\right)^0 = 135^0$

Example 6:

Convert the following measures of angles in radians.

i. 45^0 ii. 150^0 iii. 300^0 iv. $30^0 30' 30''$

Solution:

i. $45^0 = 45 \times \frac{\pi}{180} \text{ radians} = \frac{\pi}{4} \text{ radians} \approx 0.7854 \text{ radians}$

ii. $150^0 = 150 \times \frac{\pi}{180} \text{ radians}$

$= \frac{5\pi}{6} \text{ radians} \approx 2.618 \text{ radians}$

iii. $300^0 = 300 \times \frac{\pi}{180} \text{ radians}$

$= \frac{5\pi}{3} \text{ radians} \approx 5.236 \text{ radians}$

iv. $30^0 30' 30'' = 30.5083^0$

$= 30.5083 \times \frac{\pi}{180} \text{ radians}$

$\approx 30.5083 \times 0.0175 \text{ radians}$

$\approx 0.5325 \text{ radians}$

EXERCISE 6.1

1. Convert the following measure of angles in seconds.
   i. 5° ii. $30'$ iii. $10^\circ 30'$ iv. $20^\circ 20' 20''$
2. Convert the following measure of angles in minutes.
   i. 75° ii. $120''$ iii. $50^\circ 40'$ iv. $30^\circ 30' 30''$
3. Convert the following measure of angles in degrees and write the answer correct to 4 decimals places.
   i. $135'$ ii. $150''$ iii. $60^\circ 60'$ iv. $45^\circ 45' 45''$

4. Write the following measures of angles in D°M′S″.
   i. 60.125°   ii. 135.375″   iii. 60.85′   iv. 255.45°
5. Convert the following in radians. Write the answers in terms of π.
   i. 45°   ii. 150°   iii. 60°30′   iv. 120°
6. Convert the following in radians. Use $\pi = 3.1416$.
   i. 270°   ii. 60°   iii. 180°45′   iv. 75°30′45″
7. Write the following radian measures of angles in D°M′S″.
   i. $\frac{\pi}{4}$ ii. $\frac{5\pi}{6}$ iii. $\frac{\pi}{12}$ iv. $\frac{7\pi}{40}$
   v. 1   vi. 3.1416   vii. 12π viii. 5

6.2 Sector of Circle

6.2.1 Length of an Arc

Length of a circular arc and radian measure of its central angle are directly proportional. i.e., ratio between length of an arc and radian measure of its central angle remains constant.

Consider a circle of radius r with centre O. Choose an arc of length l on circumference. Let θ be the radian measure of central angle of that arc. Now choose another arc whose central angle is of 1 radian. The length (l_1) of such arc is equal to the radius. i.e. $l_1 = r$.

$$l : \theta :: l_1 : 1$$
$$l : \theta :: r : 1$$
$$l \times 1 = r \times \theta$$
$$l = r\theta$$

So, length of a circular arc is product of radius and radian measure of central angle.

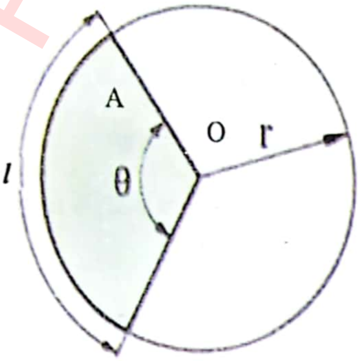


Fig. (i)

6.2.2 Area of Sector of a Circle

The union of a circle and its interior region is the circular region. The circular region bounded by an arc and its two corresponding radial segments is called sector of the circle. In figure (i) shaded region is a sector of the circle with radius r and θ the radian measure of central angle. Like arc length, area(A) of sector is directly proportional to the radian measure of central angle.

area of sector : θ : : area of circle : complete angle

$$A : \theta :: \pi r^2 : 2\pi$$
$$2\pi A = \pi r^2 \theta$$
$$A = \frac{\pi r^2 \theta}{2\pi}$$
$$= \frac{r^2\theta}{2},$$

where θ is in radians.

Do You Know?

From elementary geometry:

$$\frac{\text{Arc length}(l)}{2\pi r} = \frac{\text{Central angle of sector}(\theta)}{2\pi} = \frac{\text{Area of sector of circle }(A)}{\pi r^2}$$

$$\frac{l}{2\pi r} = \frac{\theta}{2\pi} = \frac{A}{\pi r^2}$$

$$\Rightarrow \frac{l}{2\pi r} = \frac{\theta}{2\pi} \quad \text{and} \quad \frac{\theta}{2\pi} = \frac{A}{\pi r^2}$$

$$\Rightarrow \text{Arc length}(l) = r\theta \text{ and Area of sector}(A) = \frac{1}{2}r^2\theta$$

Example 7:

Find length of an arc of a circle whose central angle and radius are given below.

i. $\theta = \frac{\pi}{4}$ rad, $r = 2$ cm   ii. $\theta = 60^0$, $r = 30$ cm

Solution:

i.
$$l = r\theta$$
$$= 2 \times \frac{\pi}{4}$$
$$= \frac{\pi}{2} \text{ cm}$$
$$= \frac{3.14}{4}$$
$$= 1.57 \text{ cm}$$

ii. $\theta = 60^0 = \frac{\pi}{3}$ rad, $r = 30$ cm

Now, $l = r\theta$
$$= 30 \times \frac{\pi}{3}$$
$$= 10\pi \text{ cm}$$
$$= 10 \times 3.14$$
$$= 31.4 \text{ cm}$$

Example 8:

Find area of the sector of a circle whose central angle is 120^0 and radius 5 cm.

Solution:

$r = 5$ cm,

$\theta = 120^0 = \frac{2\pi}{3}$ rad

$$A = \frac{r^2\theta}{2} = \frac{1}{2}(5)^2\left(\frac{2\pi}{3}\right)$$
$$= \frac{25\pi}{3} \text{ cm}^2 = 26.18 \text{ cm}^2$$

Example 9:

The length of an arc and area of sector of a circle are 4 cm and 16 cm^2 respectively. Find radius and central angle of sector.

Solution:

$l = 4$ cm, $A = 16$ cm^2, $r = ?$, $\theta = ?$

As, $l = r\theta$

$\Rightarrow r\theta = 4$

$\therefore \quad \theta = \frac{l}{r} = \frac{4}{r}$

Now $A = \frac{1}{2}r^2\theta$

$16 = \frac{1}{2}r^2 \times \frac{4}{r}$ (substituting the value of θ)

$16 = 2r$

$r = 8$ cm

$$\therefore \theta = \frac{4}{r}$$
$$= \frac{4}{8}$$
$$= \frac{1}{2} \text{ rad.}$$

EXERCISE 6.2

1. Find length of arc and area of sector for given radius and central angle.
   i. $r = 5$ cm, $\theta = \frac{\pi}{3}$ radians
   ii. $r = 12$ m, $\theta = 120^0$
   iii. $r = 6$ dm, $\theta = 60^0 45' 30''$

2. A central angle in a circle of radius 4 cm is 75^0. Find the length of the intercepted arc and area bounded by the sector.

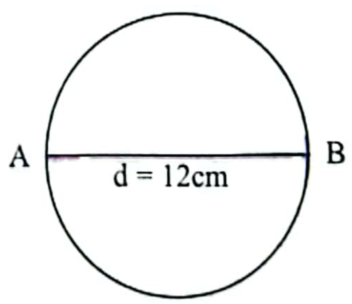
3. Find radian measure of the central angle of a circular sector for the following data.
   i. $l = 8$ cm, $r = 4$ cm
   ii. $l = 8.5$ m, $r = 2.25$ m
   iii. $A = 45$ cm^2, $r = 10.70$ cm
   iv. $A = 100$ cm^2, $l = 10$ cm

4. Find radius of circle for the following information.
   i. $l = 4$ cm, $\theta = \pi$ radians
   ii. $l = 6$ m, $\theta = 15^0$
   iii. $A = 200$ cm^2, $\theta = \frac{\pi}{4}$ radians
   iv. $A = 100$ dm^2, $l = 10$ dm

5. A 30 inch pendulum swings through an angle of 30^o. Find the length of the arc in inches through which the tip of the pendulum swings.

6. A motorcycle is traveling on a curve along a highway. The curve is an arc of a circle with radius of 10 km. If the motorcycle's speed is 42 km/h, what is the angle in degrees through which the motorcycle will turn in 21 minutes?

7. Find perimeter and area of a half circle in the following figure by using the formulae $l = r\theta,\ A = \frac{1}{2}r^2\theta$.

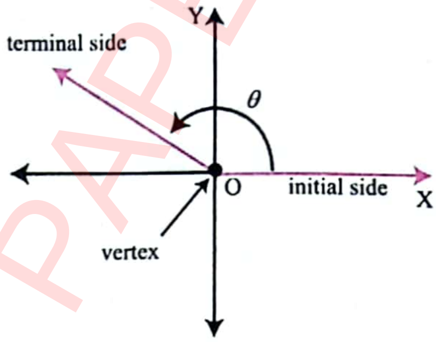


8. Find the circular measure of the angle between the hour hand and minute hand of a clock at:
   i. 9'o clock   ii. 02:30   iii. 06:45

6.3 Trigonometric Ratios

6.3.1 General Angle

In coordinate plane, the two axes divide the plane in four equivalent parts called quadrants. If we place an angle in the coordinate plane such that its vertex coincides with origin and one of its rays lies along $\overrightarrow{OX}$ (+ve x-axis), then this angle is said to be in standard position. The ray along $\overrightarrow{OX}$ is called initial ray and second ray in the plane is called terminal ray. Such an angle is called general angle.

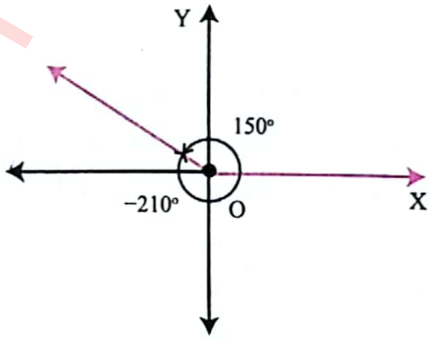


Measure of an angle is positive in anticlockwise direction and is negative in clockwise direction.

In the adjoining figure two angles in standard position are drawn. Both have same terminal ray. One of them is 150^0 (in anticlock wise direction) and the other is -210^0(in clock wise direction).

Two angles having same terminal ray in standard position are called co-terminal angles.

The measure of a general angle can be greater than 360^0.

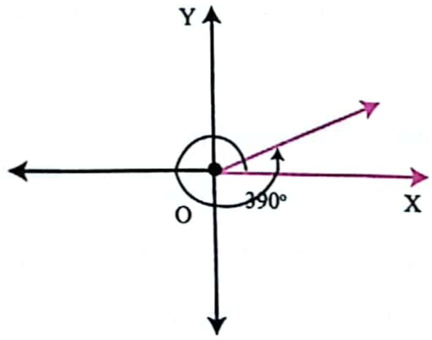


To draw a general angle whose measure is greater than 360^0, we subtract any multiple of 360^0 from its value and try to find a value less than 360^0. Terminal ray of 360^0 or that of any of its multiple coincides with 0^0 , i.e., $\overrightarrow{OX}$.

In general, two angles are co-terminal if their measures differ by some multiple of 360°. e.g., to draw an angle of 390^0, first find $390^0 - 360^0 = 30^0$, then draw 30^0 in anticlockwise direction.

390^0 and 30^0 have same terminal ray so they are co-terminal angles.

Similarly, 150^0 and -210^0 are co-terminal angles.



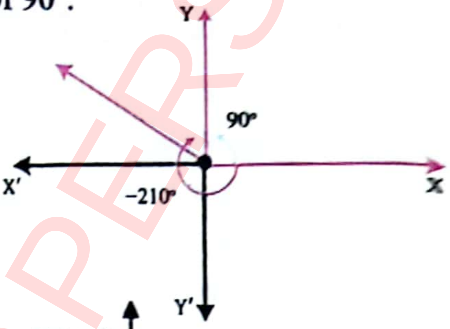
Thinking Corner!

How many co-terminal angles can be of an angle?

6.3.2 Quadrantal Angles

If terminal ray of an angle in standard position coincides with any of the axes, then it is called quadrantal angle. The measure of a quadrantal angle is a multiple of 90^0.

e.g,. $0^0, 90^0, 180^0, 270^0, 360^0, 450^0, \ldots$ are the quadrantal angles.



In radian measure, quadrantal angle is a multiple of $\frac{\pi}{2}$.

e.g., $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi, \frac{5\pi}{2}, 3\pi \ldots$ are quadrantal angles.

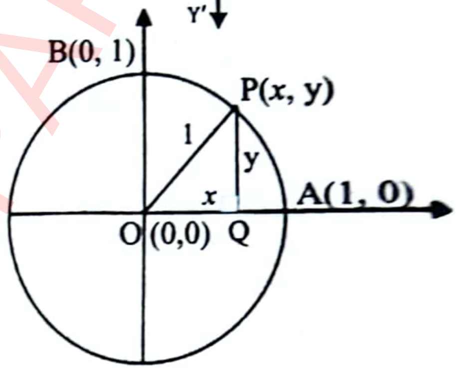
Terminal rays of $0^0, 360^0, 720^0, \ldots$ are along $\overrightarrow{OX}$.

Terminal rays of $180^0, 540^0, 900^0, \ldots$ are along $\overrightarrow{OX'}$.

Terminal rays of $90^0, 450^0, 810^0, \ldots$ are along $\overrightarrow{OY}$.

Terminal rays of $270^0, 630^0, 990^0, \ldots$ are along $\overrightarrow{OY'}$.

Unit Circle



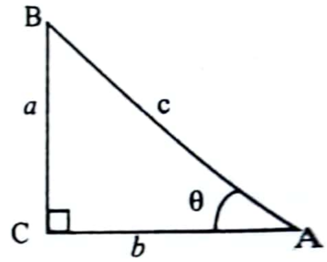
A circle whose radius is of 1 unit is called a unit circle.

In standard position centre of a unit circle is O (Origin).

The equation of a unit circle centered at O is $x^2 + y^2 = 1$.

Trigonometric Ratios

The ratio between any two sides of a right-angled triangle is called a trigonometric ratio. There are six possible trigonometric ratios corresponding to an angle.In triangle ABC, $\angle C = 90^0$, $\angle A = \theta$. AB = c, BC = a and CA = b.With respect to an acute angle θ, a is the length of perpendicular, b is the length of base and c is the length of hypotenuse.



Various trigonometric ratios are defined as:

$$\frac{Perpendicular}{Hypotenuse} = \frac{a}{c}, \frac{Base}{Hypotenuse} = \frac{b}{c}, \frac{Perpendicular}{Base} = \frac{a}{b},$$

$$\frac{Hypotenuse}{Perpendicular} = \frac{c}{a}, \frac{Hypotenuse}{Base} = \frac{c}{b}, \frac{Base}{Perpendicular} = \frac{b}{a}$$

- The ratio of perpendicular and hypotenuse is sine of θ denoted by sin θ.

  $$\therefore \sin\theta = \frac{Perpendicular}{Hypotenuse} = \frac{a}{c}$$

- The ratio of base and hypotenuse is cosine of θ denoted by cos θ.

  $$\therefore \cos\theta = \frac{Base}{Hypotenuse} = \frac{b}{c}$$

- The ratio of perpendicular and base is tangent of θ denoted by tan θ.

  $$\therefore \tan\theta = \frac{Perpendicular}{Base} = \frac{a}{b}$$

- The ratio of hypotenuse and perpendicular is cosecant θ denoted by cosec θ.

  $$\therefore \text{cosec}\,\theta = \frac{Hypotenuse}{Perpendicular} = \frac{c}{a} = \frac{1}{\sin\theta}$$

- The ratio of hypotenuse and base is secant of θ denoted by sec θ.

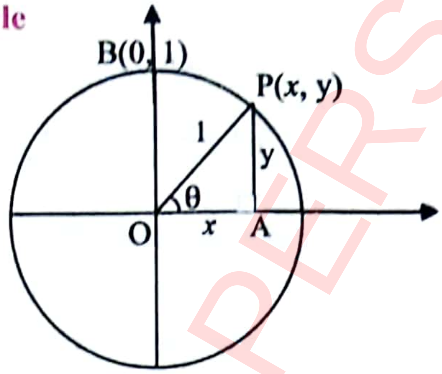
  $$\therefore \sec\theta = \frac{Hypotenuse}{Base} = \frac{c}{b} = \frac{1}{\cos\theta}$$

- The ratio of base and perpendicular is cotangent of θ denoted by cot θ.

  $$\therefore \cot\theta = \frac{Base}{Perpendicular} = \frac{b}{a} = \frac{1}{\tan\theta}$$

- cosec θ, sec θ and cot θ are respectively reciprocals of sin θ, cos θ and tan θ.

6.3.3 Trigonometric Ratios with the Help of Unit Circle



Consider a unit circle centered at O.
Mark a point $P(x, y)$ on the circle. Join P with O.
Draw PA perpendicular to x-axis.
$OA = x$, $PA = y$ and $OP = 1$(radius of circle).
If $\angle AOP = \theta$, then $\sin\theta = \frac{y}{1} = y$ and $\cos\theta = \frac{x}{1} = x$.
So, coordinates of P are $(\cos\theta, \sin\theta)$.
i.e., $x = \cos\theta$ and $y = \sin\theta$. Also, $\tan\theta = \frac{y}{x}$.

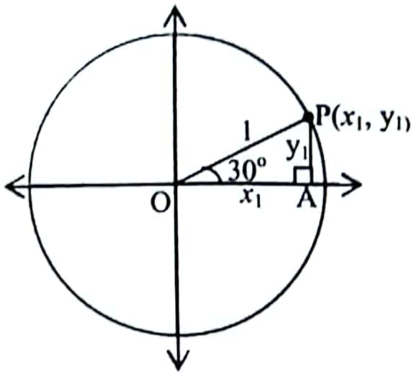
Reciprocal of y is cosec θ, reciprocal of x is sec θ and reciprocal of $\frac{y}{x}$ is cot θ.

If P is a point on a unit circle centered at O and $\angle AOP = \theta$, then coordinates of P are $(\cos\theta, \sin\theta)$.

6.3.4 Trigonometric Ratios of 30^0, 45^0 and 60^0

Trigonometric Ratios of 30^0

To find trigonometric ratios of 30^o, choose a point $P(x_1, y_1)$ on unit circle in standard position such that $\angle O = \theta = 30^o$. Draw PA perpendicular to x-axis. As in a right-angled triangle length of side opposite to 30^o is half of hypotenuse, so in right angled triangle OAP, $y_1 = \frac{1}{2}$.



Equation of unit circle is $x^2 + y^2 = 1$.
Put $x = x_1$ and $y = y_1 = \frac{1}{2}$ in above equation to get

$$x_1^2 + \left(\frac{1}{2}\right)^2 = 1$$

$$x_1^2 = 1 - \frac{1}{4}$$

$$x_1^2 = \frac{3}{4} \text{ or } x_1 = \frac{\sqrt{3}}{2} \quad (\because \text{P lies in quadrant I})$$

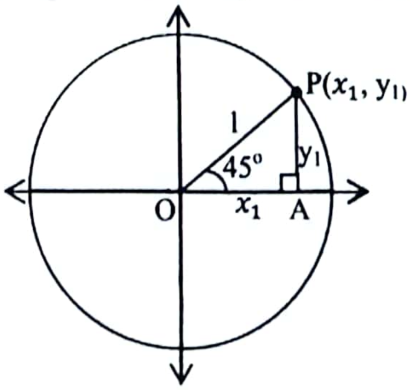
Coordinates of P are $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

So, $\cos 30^0 = x_1 = \frac{\sqrt{3}}{2}$, $\sin 30^0 = y_1 = \frac{1}{2}$, $\tan 30^0 = \frac{y_1}{x_1} = \frac{1}{\sqrt{3}}$

$\sec 30^0 = \frac{1}{x_1} = \frac{2}{\sqrt{3}}$, $\operatorname{cosec} 30^0 = \frac{1}{y_1} = 2$, and $\cot 30^0 = \frac{x_1}{y_1} = \sqrt{3}$.

Trigonometric Ratios of 45^0

To find trigonometric ratios of 45^0, choose a point $P(x_1, y_1)$ on a unit circle in standard position such that $\angle O = \theta = 45^0$. Draw PA perpendicular to x-axis. Triangle OAP is right isosceles.
Therefore, $AP = OA$ or $x_1 = y_1$.



Equation of unit circle is $x^2 + y^2 = 1$.
Put $x = x_1$ and $y = y_1 = x_1$ in above equation to get

$$x_1^2 + x_1^2 = 1$$

$$2x_1^2 = 1$$

$$x_1^2 = \frac{1}{2} \Rightarrow x_1 = \frac{1}{\sqrt{2}}, \text{ also } y_1 = \frac{1}{\sqrt{2}},$$

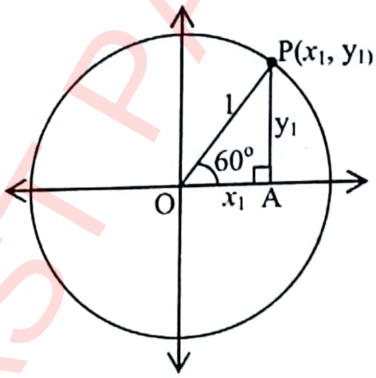
Coordinates of P are $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

So, $\cos 45^0 = x_1 = \frac{1}{\sqrt{2}}$, $\sin 45^0 = y_1 = \frac{1}{\sqrt{2}}$, $\tan 45^0 = \frac{y_1}{x_1} = 1$,

$\sec 45^0 = \frac{1}{x_1} = \sqrt{2}$, $\operatorname{cosec} 45^0 = \frac{1}{y_1} = \sqrt{2}$, and $\cot 45^0 = \frac{x_1}{y_1} = 1$.

Trigonometric Ratios of 60^0

To find trigonometric ratios of 60^0, choose a point $P(x_1, y_1)$ on unit circle in standard position such that $\angle O = \theta = 60^0$. Draw PA perpendicular to x-axis. As in a right-angled triangle two acute angles are complementary so $\angle OPA = 30^0$. As in a right-angled triangle length of side opposite to 30^0 is half of hypotenuse, so $x_1 = \frac{1}{2}$.



Equation of unit circle is $x^2 + y^2 = 1$.

Put $x = x_1 = \frac{1}{2}$ and $y = y_1$ in above equation to get

$$\left(\frac{1}{2}\right)^2 + {y_1}^2 = 1$$

$${y_1}^2 = 1 - \frac{1}{4} \Rightarrow {y_1}^2 = \frac{3}{4} \Rightarrow y_1 = \frac{\sqrt{3}}{2}, \text{ also } x_1 = \frac{1}{2},$$

Coordinates of P are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

So, $\cos 60^0 = x_1 = \frac{1}{2}$, $\sin 60^0 = y_1 = \frac{\sqrt{3}}{2}$,

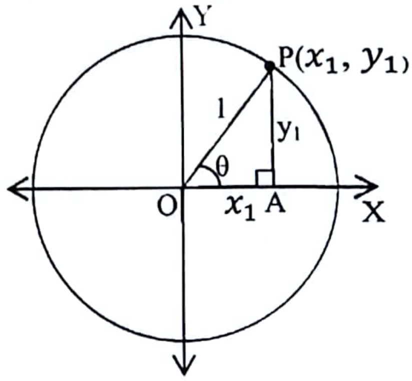
$\tan 60^0 = \frac{y_1}{x_1} = \sqrt{3}$, $\sec 60^0 = \frac{1}{x_1} = 2$,

$\operatorname{cosec} 60^0 = \frac{1}{y_1} = \frac{2}{\sqrt{3}}$, and $\cot 60^0 = \frac{x_1}{y_1} = \frac{1}{\sqrt{3}}$.

6.3.5 Signs of Trigonometric Ratios in the Four Quadrants

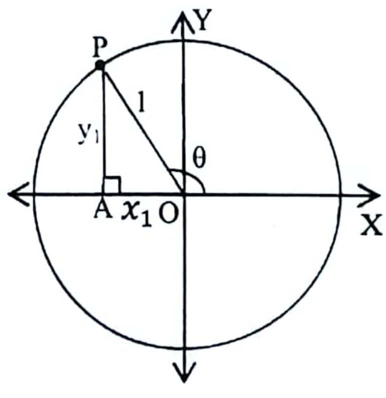
If $P(x_1, y_1)$ is a point on a unit circle such that $\angle XOP = \theta$, then $x_1 = \cos\theta$ and $y_1 = \sin\theta$.

First Quadrant



If P lies in first quadrant, then θ lies between 0^0 and 90^0. Both coordinates of P are positive i.e., $x_1 > 0$ and $y_1 > 0$. So, in first quadrant $\cos\theta > 0$ and $\sin\theta > 0$. As the quotient and reciprocals of positive numbers are also positive so in first quadrant all trigonometric ratios are positive.

Second Quadrant



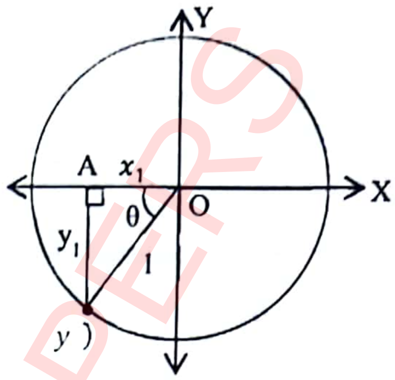
If P lies in second quadrant, then θ lies between 90^0 and 180^0. In second quadrant abscissa of a point is negative and ordinate is positive i.e. $x_1 < 0$ and $y_1 > 0$.

So, in second quadrant $\cos\theta < 0$ and $\sin\theta > 0$.

As the quotient of a negative and a positive number is negative so $\tan\theta < 0$ and $\cot\theta < 0$.

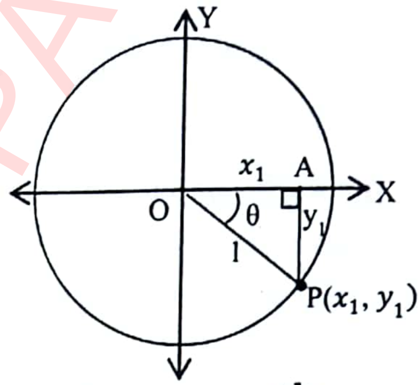
As reciprocal of a negative number is negative and that of positive number is positive so $\sec\theta < 0$ and $\operatorname{cosec}\theta > 0$.

Third Quadrant



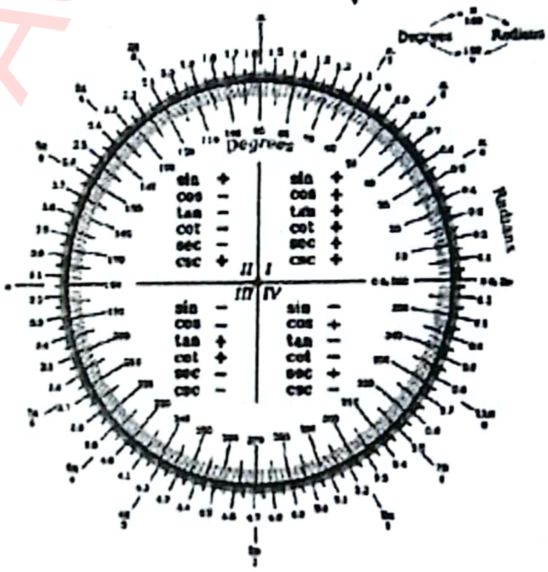
If P lies in third quadrant, then θ lies between 180^0 and 270^0. In third quadrant both abscissa and ordinate of a point are negative i.e., $x_1 < 0$ and $y_1 < 0$. So, in third quadrant cosθ < 0 and sinθ < 0. As the quotient of two negative numbers is positive so tan θ > 0 and cot θ > 0. As reciprocals of negative numbers are negative, so sec θ < 0 and cosec θ < 0.

Fourth Quadrant



If P lies in fourth quadrant, then θ lies between 270^0 and 360^0. In fourth quadrant abscissa of a point is positive and ordinate is negative i.e. $x_1 > 0$ and $y_1 < 0$.
So in fourth quadrant cos θ > 0 and sin θ < 0. As the quotient of a negative and a positive number is negative so tan θ < 0 and cot θ < 0. As reciprocal of a negative number is negative and that of positive number is positive, so sec θ > 0 and cosec θ < 0.

The figure at right would be helpful to understand the signs of trigonometric ratios in different quadrants.



To Be Observed!

When point P lies in any of the quadrants or on any of the axis, by joining it to origin, we get the angle θ formed by terminal arm OP in standard position, then:

$$\sin\theta = \frac{\textit{algebraic distance of P from x-axis}}{\textit{distance of P from origin}} = \frac{y}{r}$$

$$\cos\theta = \frac{\textit{algebraic distance of P from y-axis}}{\textit{distance of P from origin}} = \frac{x}{r}$$

$$\tan\theta = \frac{\textit{algebraic distance of P from x-axis}}{\textit{algebraic distance of P from y-axis}} = \frac{y}{x}$$

(due to the word 'algebraic', x and y will get +/− sign according to the quadrant of P)
We consider $r = 1$ in a unit circle.

6.3.6 Evaluation of Trigonometric Ratios of an Angle if One of Them is Given

We can find all the trigonometric ratios of a particular angle if one of them is given i.e. if value of 'sin θ' is given, then we can find the remaining ratios, for the same value of θ.

Example 10:

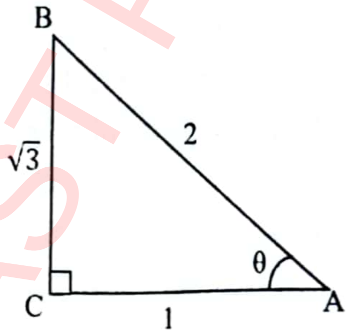
If $\sin\theta = \frac{\sqrt{3}}{2}$ and terminal ray of θ is in first quadrant, then find remaining trigonometric ratios for same value of θ.

Solution: **Method I**

Consider a right-angled triangle ABC, in which $m\angle C = 90^0$, $m\angle A = \theta$.

By definition $\sin\theta = \frac{perpendicular}{hypotenuse} = \frac{BC}{AB}$

But $\sin\theta = \frac{\sqrt{3}}{2}$ (given)



So, $BC = \sqrt{3}$ and $AB = 2$

By Pythagoras theorem $AB^2 = AC^2 + BC^2$

$$2^2 = AC^2 + (\sqrt{3})^2$$
$$4 = AC^2 + 3$$
$$4 - 3 = AC^2$$
$$AC^2 = 1 \Rightarrow AC = 1$$

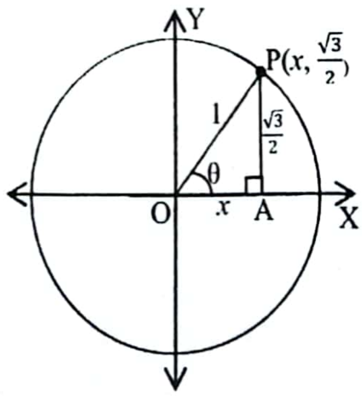
$\cos\theta = \frac{base}{hypotenuse} = \frac{AC}{AB} = \frac{1}{2}$, $\sec\theta = \frac{1}{\cos\theta} = 2$

$\tan\theta = \frac{perpendicular}{base} = \frac{BC}{AC} = \frac{\sqrt{3}}{1} = \sqrt{3}$, $\cot\theta = \frac{1}{\tan\theta} = \frac{1}{\sqrt{3}}$, $\csc\theta = \frac{1}{\sin\theta} = \frac{2}{\sqrt{3}}$

Method II

Consider a unit circle at O. Mark a point P on the circle in first quadrant whose y coordinate is $\frac{\sqrt{3}}{2}$.

The equation of unit circle is $x^2 + y^2 = 1$.



Put $y = \frac{\sqrt{3}}{2}$

$$x^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = 1$$
$$x^2 = 1 - \frac{3}{4} = \frac{1}{4} \Rightarrow x = \pm\frac{1}{2}$$

Since, P is in first quadrant so $x > 0$. i.e., $x = \frac{1}{2}$

So, $\cos\theta = x = \frac{1}{2}$

$\tan\theta = \frac{y}{x} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$, $\csc\theta = \frac{1}{\sin\theta} = \frac{2}{\sqrt{3}}$

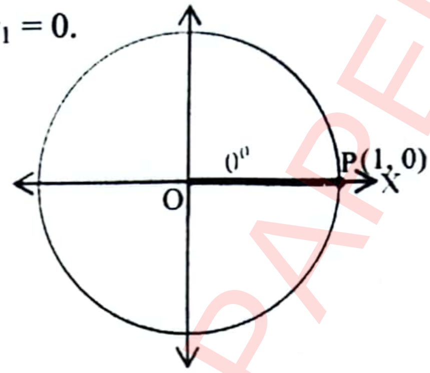
$\sec\theta = \frac{1}{\cos\theta} = 2$, $\cot\theta = \frac{1}{\tan\theta} = \frac{1}{\sqrt{3}}$

6.3.7 Trigonometric Ratios of Quadrantal Angles

Trigonometric Ratios of 0^0

To find trigonometric ratios of 0^0, choose a point $P(x_1, y_1)$ on unit circle in standard position such that $\angle XOP = \theta = 0^0$.

As P lies on x-axis so its coordinates are $x_1 = 1$ and $y_1 = 0$.



So, $\cos 0^0 = x_1 = 1$

$\sin 0^0 = y_1 = 0$

$\tan 0^0 = \frac{y_1}{x_1} = \frac{0}{1} = 0$

$\sec 0^0 = \frac{1}{x_1} = \frac{1}{1} = 1$

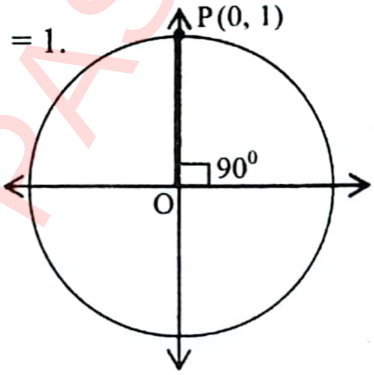
$\operatorname{cosec} 0^0 = \frac{1}{y_1} = \frac{1}{0} = \infty$ (undefined)

$\cot 0^0 = \frac{x_1}{y_1} = \frac{1}{0} = \infty$ (undefined)

Trigonometric Ratios of 90^0

To find trigonometric ratios of 90^0, choose a point $P(x_1, y_1)$ on unit circle in standard position such that $\angle XOP = \theta = 90^0$.

As P lies on y-axis so its coordinates are $x_1 = 0$ and $y_1 = 1$.



So, $\cos 90^0 = x_1 = 0$

$\sin 90^0 = y_1 = 1$

$\tan 90^0 = \frac{y_1}{x_1} = \frac{1}{0} = \infty$ (undefined)

$\sec 90^0 = \frac{1}{x_1} = \frac{1}{0} = \infty$ (undefined)

$\operatorname{cosec} 90^0 = \frac{1}{y_1} = \frac{1}{1} = 1,$

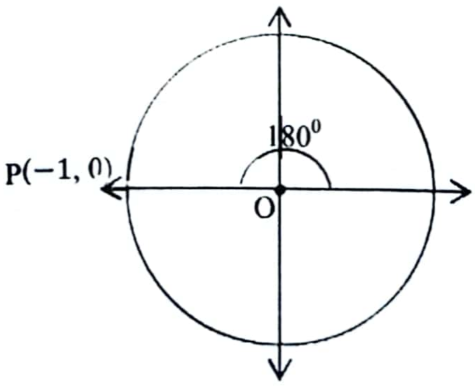
$\cot 90^0 = \frac{x_1}{y_1} = \frac{0}{1} = 0$

Trigonometric Ratios of 180^0

To find trigonometric ratios of 180^0, choose a point $P(x_1, y_1)$ on unit circle in standard position such that $\angle XOP = \theta = 180^0$.

As P lies on x-axis at left side of O so its coordinates are

$x_1 = -1$ and $y_1 = 0$.



So, $\cos 180^0 = x_1 = -1$

$\sin 180^0 = y_1 = 0$

$\tan 180^0 = \frac{y_1}{x_1} = \frac{0}{-1} = 0$

$\sec 180^0 = \frac{1}{x_1} = \frac{1}{-1} = -1$

$\operatorname{cosec} 180^0 = \frac{1}{y_1} = \frac{1}{0} = \infty$ (undefined)

$\cot 180^0 = \frac{x_1}{y_1} = \frac{-1}{0} = \infty$ (undefined)

Trigonometric Ratios of 270^0

To find trigonometric ratios of 270^0, choose a point $P(x_1, y_1)$ on unit circle in standard position such that $\angle XOP = \theta = 270^0$.

As P lies on y-axis below origin, therefore its coordinates are $x_1 = 0$ and $y_1 = -1$.

So, $\cos 270^0 = x_1 = 0$

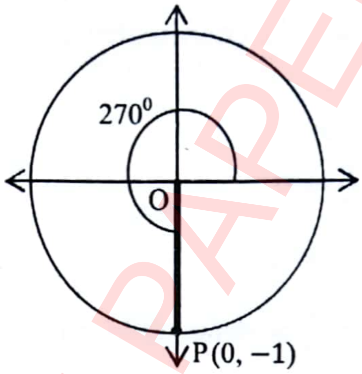
$\sin 270^0 = y_1 = -1$

$\tan 270^0 = \frac{y_1}{x_1} = \frac{-1}{0} = \infty$ (undefined)

$\sec 270^0 = \frac{1}{x_1} = \frac{1}{0} = \infty$ (undefined)

$\text{cosec } 270^0 = \frac{1}{y_1} = \frac{1}{-1} = -1$

$\cot 270^0 = \frac{x_1}{y_1} = \frac{0}{-1} = 0$



Trigonometric Ratios of 360^0

To find trigonometric ratios of 360^0, choose a point $P(x_1, y_1)$ on unit circle in standard position such that $\angle XOP = \theta = 360^0$. It coincides with 0^0.

As P lies on x-axis so its coordinates are $x_1 = 1$ and $y_1 = 0$.

So, $\cos 360^0 = x_1 = 1$

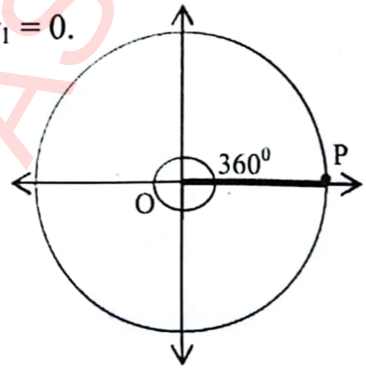
$\sin 360^0 = y_1 = 0$

$\tan 360^0 = \frac{y_1}{x_1} = \frac{0}{1} = 0$

$\sec 360^0 = \frac{1}{x_1} = \frac{1}{1} = 1$

$\text{cosec } 360^0 = \frac{1}{y_1} = \frac{1}{0} = \infty$ (undefined)

$\cot 360^0 = \frac{x_1}{y_1} = \frac{1}{0} = \infty$ (undefined)



EXERCISE 6.3

1. Complete the following table.

θ	0^0	30^0	45^0	60^0	90^0	180^0	270^0	360^0
$\sin\theta$	0			$\frac{\sqrt{3}}{2}$			-1	
$\cos\theta$						-1		
$\tan\theta$			1					
$\text{cosec }\theta$	∞					∞		
$\sec\theta$				2				1
$\cot\theta$		$\sqrt{3}$			0			

2. Evaluate the following.

i. $\sin 60^0 - \cos 30^0$ ii. $\sec 45^0 \operatorname{cosec} 45^0 - \sin 30^0 \cos 60^0$

iii. $\dfrac{\tan\frac{\pi}{3} - \tan\frac{\pi}{6}}{1 + \tan\frac{\pi}{3}\tan\frac{\pi}{6}}$ iv. $\cos^2\frac{\pi}{3} - \sin^2\frac{\pi}{3}$ v. $\sin\frac{\pi}{2}\cos\frac{\pi}{3} - \cos\frac{\pi}{2}\sin\frac{\pi}{3}$

3. For $\theta = \frac{\pi}{6}$, verify the following statements.

i. $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$ ii. $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

iii. $\cos 2\theta = 1 - 2\sin^2\theta$ iv. $\sec^2\theta - \tan^2\theta = 1$ v. $\operatorname{cosec}^2\theta = 1 + \cot^2\theta$

4. Write value of θ for the following ratios $(0 < \theta < 2\pi)$.

i. $\sin\theta = \frac{\sqrt{2}}{2}$ ii. $\operatorname{cosec}\theta = -1$ iii. $\tan\theta = 1$

iv. $\sec\theta = \frac{\sqrt{12}}{3}$ v. $\cos\theta = 0$

5. If terminal ray of θ is in first quadrant and $\cos\theta = \frac{1}{2}$, then find remaining trigonometric ratios of θ.

6. If terminal ray of θ is in second quadrant and $\operatorname{cosec}\theta = 2$, then find the value of $\cos\theta\left(\dfrac{\sin^2\theta - \cos^2\theta}{\sin^2\theta + \cos^2\theta}\right)$.

7. If $\tan\theta = -1$, and terminal ray of θ is not in second quadrant, then find remaining trigonometric ratios of θ.

6.4 Trigonometric Identities

Identity is an equation which remains true for all possible replacements of variable involved in it. e.g. $(x + 1)(x - 1) = x^2 - 1$ is an identity.

Trigonometric identity is an identity involving different trigonometric ratios in it. Trigonometric identities are of many types. Some of them are given below.

i. Reciprocal identities
ii. Quotient identities
iii. Pythagorean identities

6.4.1 Reciprocal Identities

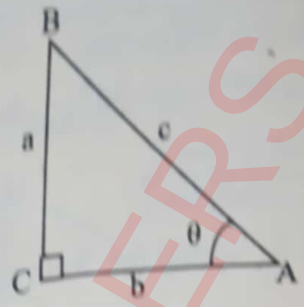
Cosec θ, sec θ and cot θ are reciprocals of sin θ, cos θ and tan θ respectively.

i.e. $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$, $\sec\theta = \dfrac{1}{\cos\theta}$, and $\cot\theta = \dfrac{1}{\tan\theta}$

Proves of these identities are left for students.

6.4.2 Quotient Identities

i. $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ ii. $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$



Proof:

Consider a right-angled triangle ABC, in which $\angle C = 90^0$, $\angle A = \theta$, AB = c, BC = a and AC = b.

By definition, $\sin\theta = \frac{a}{c}$, $\cos\theta = \frac{b}{c}$ and $\tan\theta = \frac{a}{b}$.

$\tan\theta = \frac{a}{b} = \frac{\frac{a}{c}}{\frac{b}{c}}$ (dividing numerator and denominator by c)

$= \dfrac{\sin\theta}{\cos\theta}$

Similarly, we can prove the second quotient identity.

6.4.3 Pythagorean Identities

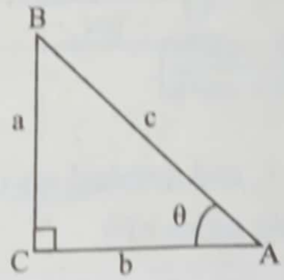
i. $\sin^2\theta + \cos^2\theta = 1$

ii. $\sec^2\theta - \tan^2\theta = 1$

iii. $\text{cosec}^2\theta - \cot^2\theta = 1$

Proof:

Consider a right-angled triangle ABC,
in which $\angle C = 90^0$, $\angle A = \theta$, AB = c, BC = a and AC = b.



i. $\sin^2\theta + \cos^2\theta = 1$

By definition, $\sin\theta = \frac{a}{c}$ and $\cos\theta = \frac{b}{c}$

By Pythagoras theorem, $a^2 + b^2 = c^2$

Dividing this Pythagorean equation by c^2 we get

$$\frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{c^2}{c^2}$$

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$$

$(\sin\theta)^2 + (\cos\theta)^2 = 1$ or $\sin^2\theta + \cos^2\theta = 1$ $(\because (\sin\theta)^2 = \sin^2\theta)$

ii. $\sec^2\theta - \tan^2\theta = 1$

By definition, $\sec\theta = \frac{c}{b}$ and $\tan\theta = \frac{a}{b}$

By Pythagoras theorem, $a^2 + b^2 = c^2$

Dividing this Pythagorean equation by b^2 we get

$$\frac{a^2}{b^2} + \frac{b^2}{b^2} = \frac{c^2}{b^2}$$

$$\left(\frac{a}{b}\right)^2 + 1 = \left(\frac{c}{b}\right)^2$$

$$(\tan\theta)^2 + 1 = (\sec\theta)^2$$

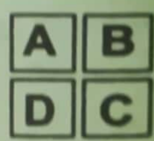
$\tan^2\theta + 1 = \sec^2\theta$ or $\sec^2\theta - \tan^2\theta = 1$

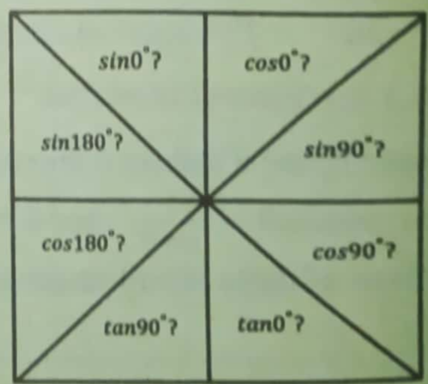
iii. $\text{cosec}^2\theta - \cot^2\theta = 1$

By definition, $\text{cosec}\,\theta = \frac{c}{a}$ and $\cot\theta = \frac{b}{a}$

Funmatics

Help students to make such a cootie catcher to play with friends.





By Pythagoras theorem, $a^2 + b^2 = c^2$

Dividing this Pythagorean equation by a^2 we get

$$\frac{a^2}{a^2} + \frac{b^2}{a^2} = \frac{c^2}{a^2}$$

$$1 + \left(\frac{b}{a}\right)^2 = \left(\frac{c}{a}\right)^2$$

$$1 + (\cot\theta)^2 = (\operatorname{cosec}\theta)^2$$

$$1 + \cot^2\theta = \operatorname{cosec}^2\theta$$

or $\operatorname{cosec}^2\theta - \cot^2\theta = 1$

Example 11:

Use basic trigonometric identities to prove the following relations.

i. $\cos\alpha + \sin\alpha\tan\alpha = \sec\alpha$

ii. $\dfrac{\tan\alpha}{\sec\alpha - 1} = \dfrac{\sec\alpha + 1}{\tan\alpha}$

iii. $\dfrac{\sin^2\alpha + 2\cos\alpha - 1}{\sin^2\alpha + 3\cos\alpha - 3} = \dfrac{1}{1 - \sec\alpha}$

Solution:

i. L.H.S. $= \cos\alpha + \sin\alpha\tan\alpha$

$$= \cos\alpha + \sin\alpha\frac{\sin\alpha}{\cos\alpha}$$

$$= \cos\alpha + \frac{\sin^2\alpha}{\cos\alpha}$$

$$= \frac{\cos^2\alpha + \sin^2\alpha}{\cos\alpha}$$

$$= \frac{1}{\cos\alpha} = \sec\alpha = \text{R.H.S.}$$

Hence, $\cos\alpha + \sin\alpha\tan\alpha = \sec\alpha$

ii. L.H.S. $= \dfrac{\tan\alpha}{\sec\alpha - 1}$

$$= \frac{\tan\alpha}{\sec\alpha - 1} \times \frac{\sec\alpha + 1}{\sec\alpha + 1}$$

$$= \frac{\tan\alpha(\sec\alpha + 1)}{\sec^2\alpha - 1}$$

$$= \frac{\tan\alpha(\sec\alpha + 1)}{\tan^2\alpha}$$

$$= \frac{\sec\alpha + 1}{\tan\alpha} = \text{R.H.S.}$$

Hence, $\dfrac{\tan\alpha}{\sec\alpha - 1} = \dfrac{\sec\alpha + 1}{\tan\alpha}$

iii. L.H.S. $= \dfrac{\sin^2\alpha + 2\cos\alpha - 1}{\sin^2\alpha + 3\cos\alpha - 3}$

$$= \frac{1 - \cos^2\alpha + 2\cos\alpha - 1}{1 - \cos^2\alpha + 3\cos\alpha - 3}$$

$$= \frac{-\cos^2\alpha+2\cos\alpha}{-\cos^2\alpha+3\cos\alpha-2}$$

$$= \frac{-\cos\alpha(\cos\alpha-2)}{-(\cos^2\alpha-3\cos\alpha+2)}$$

$$= \frac{\cos\alpha(\cos\alpha-2)}{\cos^2\alpha-2\cos\alpha-\cos\alpha+2}$$

$$= \frac{\cos\alpha(\cos\alpha-2)}{\cos\alpha(\cos\alpha-2)-1(\cos\alpha-2)}$$

$$= \frac{\cos\alpha(\cos\alpha-2)}{(\cos\alpha-1)(\cos\alpha-2)} = \frac{\cos\alpha}{\cos\alpha-1} = \frac{\frac{\text{Cos}\,\alpha}{\text{Cos}\,\alpha}}{\frac{\text{Cos}\,\alpha-1}{\text{Cos}\,\alpha}}$$

$$= \frac{1}{\frac{\cos\alpha}{\cos\alpha}-\frac{1}{\cos\alpha}} = \frac{1}{1-\sec\alpha} = \text{R.H.S.}$$

Hence $\frac{\text{Sin}^2\alpha+2\text{Cos}\,\alpha-1}{\text{Sin}^2\alpha+3\text{Cos}\,\alpha-3} = \frac{1}{1-\text{Sec}\,\alpha}$

Brain Buster

Check why

$\text{Sin}30^\circ \cos 30^\circ \cos 90^\circ \sin 90 = 0$

Example 12:

Solve the following triangles. (Find unknown elements of the following triangles)

i. ΔABC, $\angle C = 90^0$, $\angle A = 30^0$, a = 6 cm

ii. ΔABC, $\angle C = 90^0$, c = 12 cm, b = $6\sqrt{2}$ cm

i. Given Elements:

ΔABC, $\angle C = 90^0$, $\angle A = 30^0$, a = 6 cm

Required Elements:

$\angle B$, b, c.

Solution:

In ΔABC, $\angle A + \angle B + \angle C = 180^0$

$30^0 + \angle B + 90^0 = 180^0$

$\angle B = 180^0 - 90^0 - 30^0$

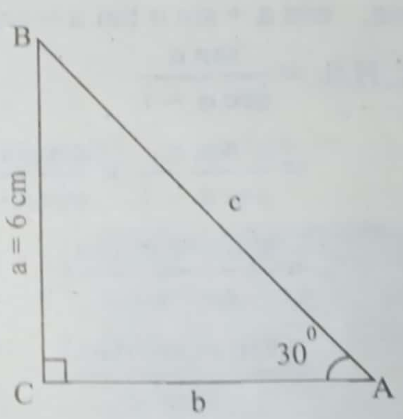
$\angle B = 60^0$

$\sin\theta = \frac{\text{Perpendicular}}{\text{Hypotenuse}}$

$\sin 30^0 = \frac{a}{c}$

$\frac{1}{2} = \frac{6}{c}$

c = 12 cm



In ΔABC, by Pythagoras theorem,

$c^2 = a^2 + b^2$

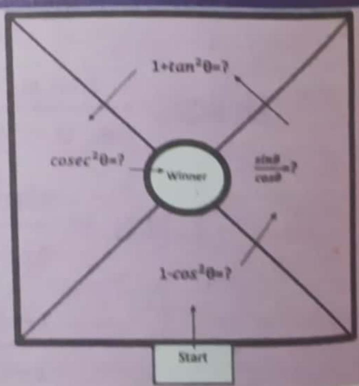
$12^2 = 6^2 + b^2$

$144 = 36 + b^2$

$b^2 = 144 - 36$

Math Play Ground

1. Make a hopscotch as given in the playground.
2. Ask a student to start hopping and finding answers.



$$b^2 = 108$$
$$b = \sqrt{108} = 6\sqrt{3}\text{ cm}$$

Therefore, required elements are $\angle B = 60^0$, $b = 6\sqrt{3}$ cm, $c = 12$ cm.

ii. Given Elements:

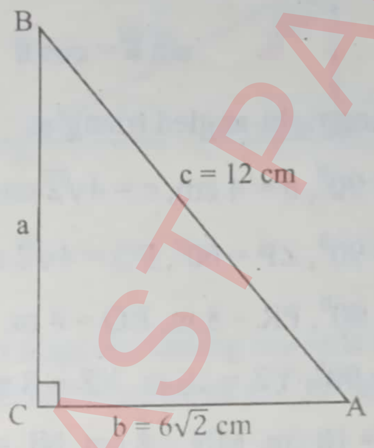
ΔABC, $\angle C = 90^0$, $c = 12$ cm, $b = 6\sqrt{2}$ cm

Required Elements:

$\angle A$, $\angle B$, a

Solution: In ΔABC, by Pythagoras theorem,

$$c^2 = a^2 + b^2$$
$$12^2 = a^2 + (6\sqrt{2})^2$$
$$144 = a^2 + 72$$
$$a^2 = 144 - 72$$
$$a^2 = 72$$
$$a = \sqrt{72} = 6\sqrt{2}\text{ cm}$$



$$\cos\theta = \frac{\text{Base}}{\text{Hypotenuse}}$$
$$\cos\angle A = \frac{b}{c}$$
$$\cos\angle A = \frac{6\sqrt{2}}{12} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$
$$\angle A = 45^0$$

In ΔABC, $\angle A + \angle B + \angle C = 180^0$

$$45^0 + \angle B + 90^0 = 180^0$$
$$\angle B = 180^0 - 90^0 - 45^0 \Rightarrow \angle B = 45^0$$

Therefore, required elements are $\angle A = 45^0$, $\angle B = 45^0$, $a = 6\sqrt{2}$ cm.

EXERCISE 6.4

1. Prove the following relations by using basic trigonometric identities.

i. $(1 - \sin^2\theta)\sec^2\theta = 1$

ii. $\frac{1-\sin\theta}{1+\sin\theta} = (\sec\theta - \tan\theta)^2$

iii. $\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{\sin\theta}{1+\cos\theta}$

iv. $\frac{\sin\theta - 2\sin^3\theta}{2\cos^3\theta - \cos\theta} = \tan\theta$

v. $\sqrt{\sec^2\theta + \operatorname{cosec}^2\theta} = \tan\theta + \cot\theta$

vi. $\frac{1}{\sec\theta - \tan\theta} = \sec\theta + \tan\theta$

vii. $\sin^2\theta + \cos^4\theta = \sin^4\theta + \cos^2\theta$

viii. $\frac{\tan\theta+\sin\theta}{\tan\theta-\sin\theta} = \frac{\sec\theta+1}{\sec\theta-1}$

ix. $\sec^2\theta + \text{cosec}^2\theta = \sec^2\theta\,\text{cosec}^2\theta$

x. $\sin^6\theta + \cos^6\theta = 1 - 3\sin^2\theta\cos^2\theta$

2. If $x\cos\theta + y\sin\theta = m$ and $x\sin\theta - y\cos\theta = n$, prove that $x^2 + y^2 = m^2 + n^2$.

3. Find values of θ for the following equations in the interval $0 \le \theta \le \frac{\pi}{2}$.

i $2\sin^2\theta = \frac{1}{2}$ ii. $\sin\theta = \cos\theta$ iii. $\sec^2\theta - 2\tan^2\theta = 0$

4. Solve the following right-angled triangles

i. ΔABC, $\angle C = 90^0$, $a = 4$ cm, $c = 4\sqrt{2}$ cm.

ii. ΔPQR, $\angle Q = 90^0$, $\angle P = 60^0$, $PQ = 4\sqrt{3}$ cm.

iii. ΔPQR, $\angle R = 90^0$, $PR = 8$ m, $RQ = 8$ m.

iv. ΔXYZ, $\angle X = 90^0$, $YZ = 16$ m, $XZ = 8$ m.

v. ΔLMN, $LM = 10$ cm, $MN = 8$ cm, $NL = 6$ cm.

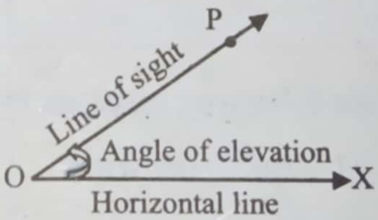
5. If $x = r\cos\alpha\sin\beta$, $y = r\cos\alpha\cos\beta$ and $z = r\sin\alpha$, then find value of $x^2 + y^2 + z^2$.

6.5 Angles of Elevation and Depression

Line of Sight

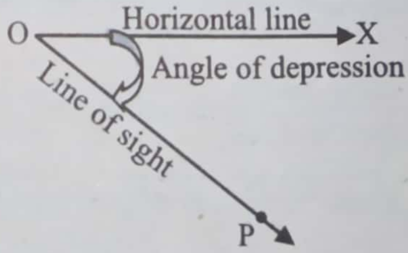
If an object P is observed from a point of observation O, then the line OP is called **line of sight.**

Angle of Elevation



If O is point of observation, OX is a horizontal line (ray)and an object P lies above the line OX, then the angle between the horizontal line and the line of sight to an object above the horizontal line is called angle of elevation . In the adjoining figure $\angle XOP$ is angle of elevation.

Angle of Depression



If O is point of observation, OX is a horizontal line (ray) and an object P lies below the line OX, then angle between the horizontal line and the line of sight to an object below the horizontal line is called angle of depression. In the adjoining figure $\angle XOP$ is angle of depression.

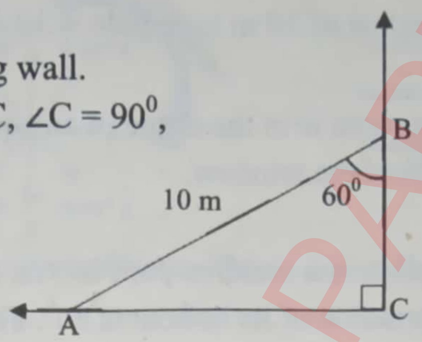
Applications

Example 13:

A ladder 10 m long leaning against a vertical wall makes an angle of 60^0 with wall. Find height of ladder along the wall.

Solution:

Let AB is a ladder. BC is height of ladder along wall.
AC is along horizontal ground. In triangle ABC, $\angle C = 90^0$,
$\angle B = 60^0$ and AB = 10 m.



$$\cos 60^0 = \frac{BC}{AB}$$
$$\frac{1}{2} = \frac{BC}{10}$$
$$BC = 5 \text{ m}$$

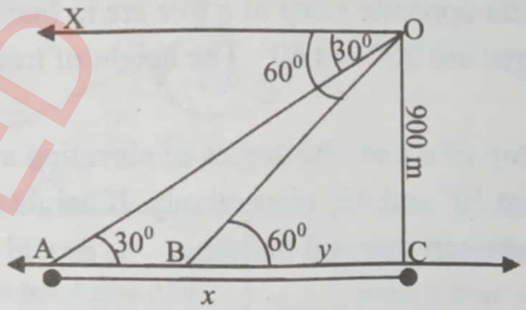
So, height of ladder along wall = 10 m.

Example 14:

An aeroplane at an altitude of 900 m finds that two ships are sailing towards it in the same direction. The angles of depression of ships as observed from plane are 30^0 and 60^0. Find distance between ships.

Solution:

In the figure O is position of plane. A and B are two ships. $\angle XOA$ and $\angle XOB$ are angles of depression. As alternate angles of parallel lines are equal.
So, $\angle A = 30^0$ and $\angle B = 60^0$
Let AC = x and BC = y.
In right angled triangle ACO,



$$\tan 30^0 = \frac{900}{x}$$
$$\frac{1}{\sqrt{3}} = \frac{900}{x}$$
$$x = 900\sqrt{3} = 1558.85 \text{ m}$$

In right angled triangle BCO,

$$\tan 60^0 = \frac{900}{y}$$
$$\sqrt{3} = \frac{900}{y} \Rightarrow y = \frac{900}{\sqrt{3}} = 519.62 \text{ m}$$

Distance between ships $= x - y$
$= 1558.85 - 519.62$
$= 1039.23$ m

Check Point

One acute angle of a right triangle is a. Find both acute angles if $\sin a = \cos a$.

EXERCISE 6.5

1. From a point at a distance of 20 m from a tree, angle of elevation of top of a tree is 30^0. Find height of the tree.

2. The length of shadow of 10 m high pole is $10\sqrt{3}$ m. Find angle of elevation of the sun.

3. The window of a room is in the shape of an equilateral triangle. The length of each side is 1.6m. Find height of the window.

4. The height of a slide in a children park is 5 m. A girl standing at the top of the slide observes that angle of depression of its bottom is 30^0. Find length of the slide.

5. Two pillars of equal height stand on either side of a roadway which is 120 m wide. At a point on the road between pillars, the angles of elevation of the pillars are 60^0 and 30^0. Find height of each pillar and position of the point.

6. From the top of a tower of height 120 m, angles of depression of two boats on same side of tower at water level are 60^o and 45^o. Find distance between the boats.

7. Two men on opposite sides of a tree are in line with it. They observe that angles of elevation of top of tree are 30^o and 60^o. The height of tree is 15 m. Find distance between the men.

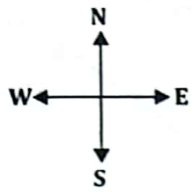
8. From the top of a tree, the angles of elevation and depression of the top and bottom of a building are 30^0 and 45^0 respectively. If height of tree is 12 m, find height of building and distance between tree and building.

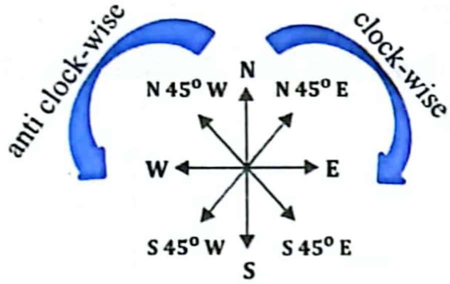
9. From a point A on ground at a distance of 200m. angle of elevation of top of a tower is α. There is another point B 80m nearer to the tower. The angle of elevation of top of tower from B is β. If $\tan\alpha = \frac{2}{5}$, find height of tower and value of β.

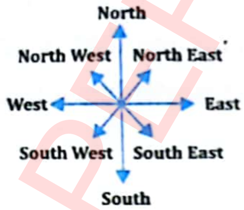
10. A boat moving away from a light house 206.6 m high, takes 120 seconds to change the angle of elevation of top of light house from 60^o to 45^o. Find the speed of boat.

6.6 Bearing

When we want to move from one position to another, we need to know not only the distance to be traveled but also the direction of movement. Direction sense is the ability to know a person's current location and then find the correct way towards the destination. It helps us in getting clear picture of bearing.







The four cardinal directions are north, east, south and west. These are marked by the initials N, E, S and W. Starting from north with an angle of zero degree, east is at right angle to north and intercardinal direction north-east is at an angle of 45 degrees to north. Similarly south- east is at angle of 135 degrees to north and so on.

Bearings are the angles measured clockwise from due north and are expressed as three digit numbers. Bearings are mostly used by sailors and pilots to describe the direction they are travelling and to navigate from one point to another.

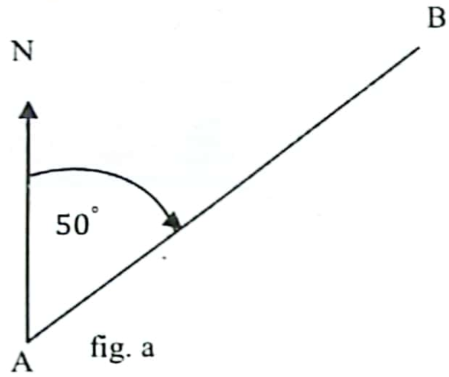


fig. a

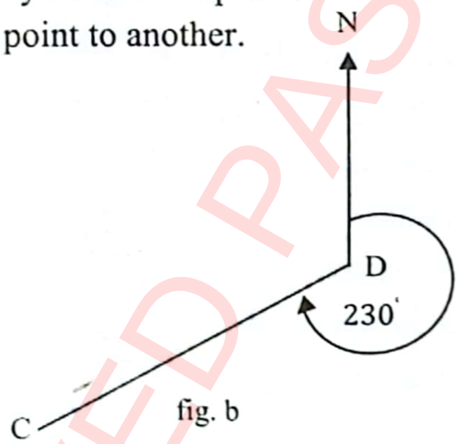


fig. b

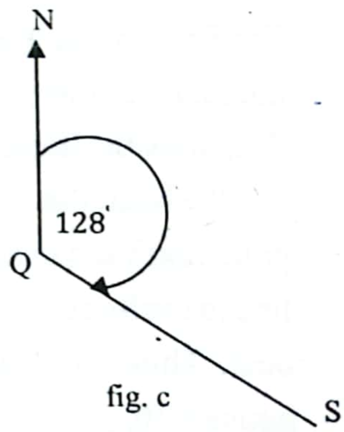


fig. c

In fig.a, the bearing of point B from point A is 050°.

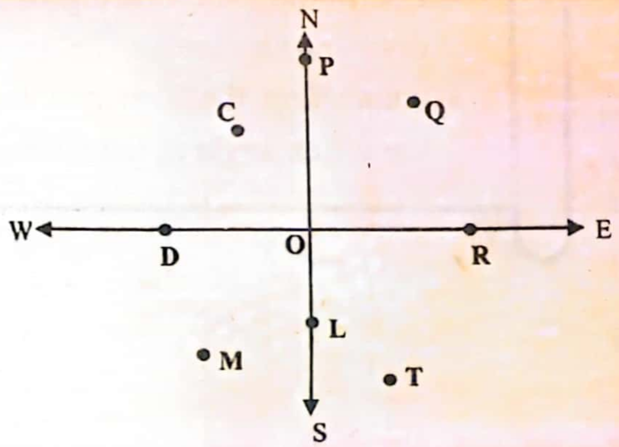
In fig.b, the bearing of point C from point D is 230°.

In fig.c, the bearing of point S from point Q is 128°.

Above figures are not drawn upto scale.

Check Point

Join all the marked points to O, then find their bearing with the help of protractor. Also write the type of angle in each case. What is the bearing of north, south, east and west direction lines?



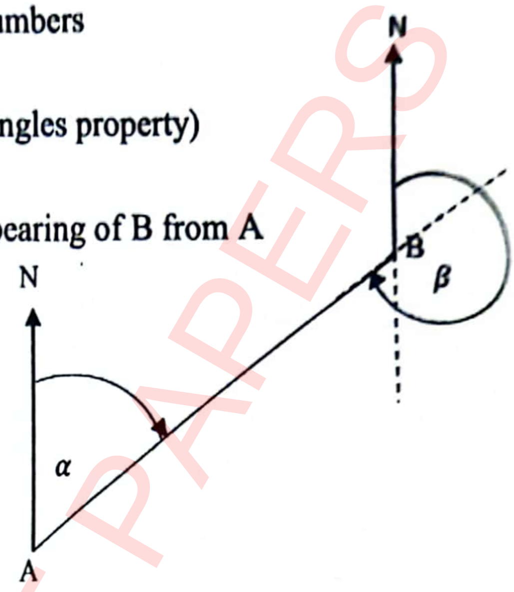
Points to ponder!

- Bearing is an angle expressed with three-digit numbers measured in clockwise direction from north.
- If α is acute, then $\beta = 180^\circ + \alpha$ (corresponding angles property)

$$\Rightarrow \alpha = \beta - 180^\circ$$

This means that bearing of A from B = 180° + bearing of B from A

- Exact north has a bearing of 0°.



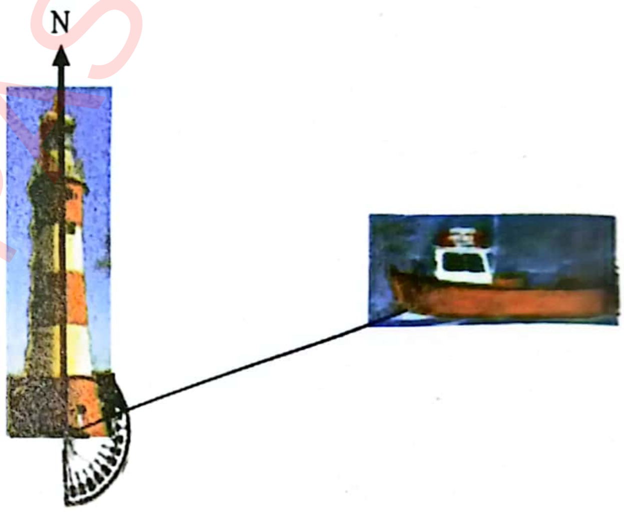
Example 15:

Find the bearing of boat from lighthouse in the given figure.

Solution:

For finding the bearing of boat from light house, first draw the north line at light house, then draw the line connecting the light house and the boat. Place your protractor with zero point north and measure the angle from north line to the line connecting light house and the boat. Thus, the bearing of boat from light house = 70°.

(The figure is not drawn accurately)



Thinking Corner!

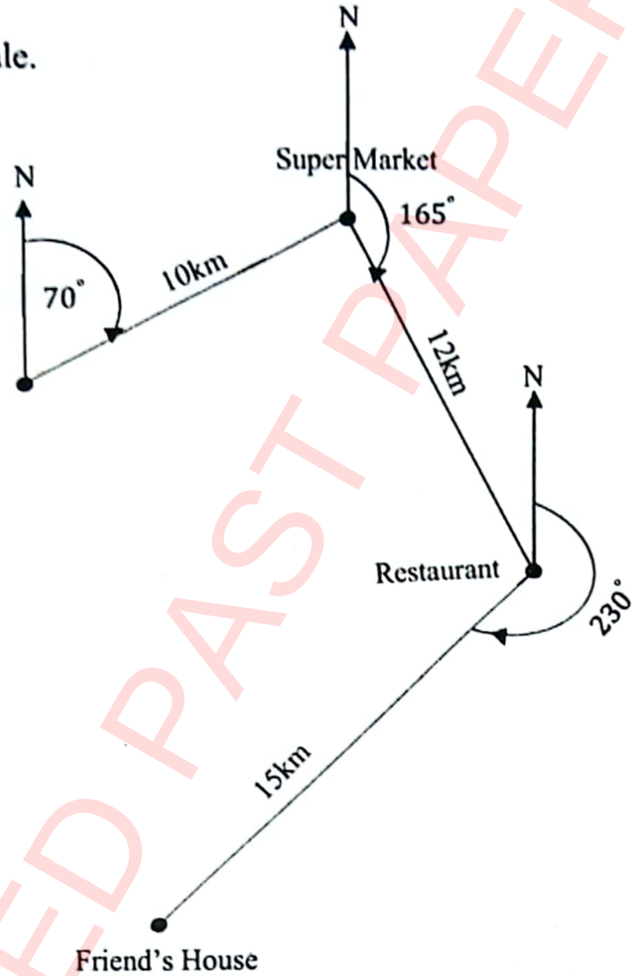
For finding the bearing of light house from boat, draw north line at the boat, then draw the line connecting the boat and the light house. Measure the reflex angle to get the required bearing.

Example 16:

Arsalan travelled at a bearing of 70° for 10km to reach super market. Then, he travelled at a bearing of 165° for 12km to reach a restaurant. Finally, he travelled at a bearing of 230° for 15km to reach his friend's house. Make a labelled illustration to show his journey.

Solution:

In the figure, angles are not marked upto scale.



We can write bearing of 070° from starting point to supermarket as N 70°E.

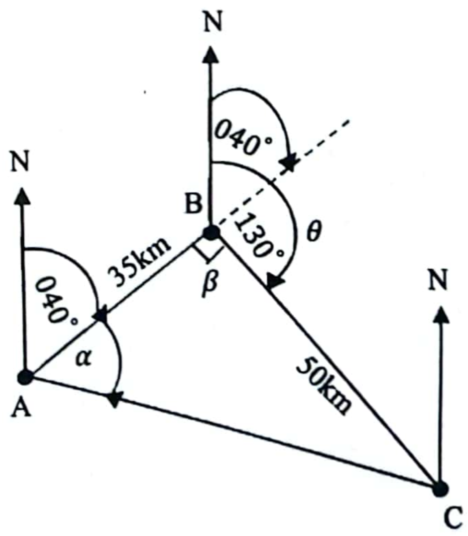
Example 17:

A ship sails from port A to port B at a bearing of 040° for 35 km. Then, it sails at a bearing of 130° for 50 km from port B to port C. Find:

(a) the distance between port A and port C

(b) the bearing of C from A

Solution:



(a) By drawing the path according to given information, we observe that $\theta = 130^\circ - 40^\circ = 90^\circ$(since two norths at A and B are parallel).

Therefore, $\beta = 180^\circ - 90^\circ = 90^\circ$.

Thus, triangle ABC is right with right angle at B.

In right ΔABC by using Pythagoras theorem,

$$m\overline{AC} = \sqrt{(m\overline{AB})^2 + (m\overline{BC})^2}$$

$$= \sqrt{(35)^2 + (50)^2} = \sqrt{3725} \approx 61.03 \text{ km}$$

(b) Bearing of C from A = $040^\circ + \alpha$

First we find α in right ΔABC,

$$\tan\alpha = \frac{m\overline{BC}}{m\overline{AB}}$$

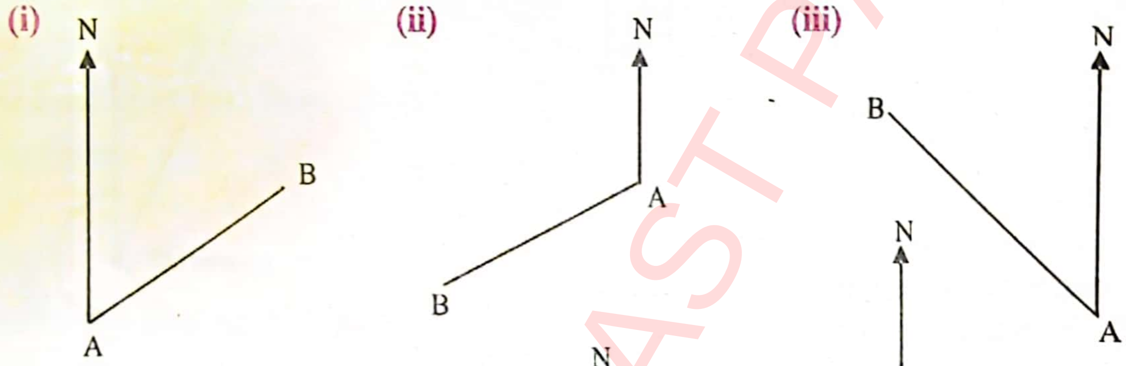
$$= \frac{50}{35} \approx 1.4286$$

$$\Rightarrow \alpha = \tan^{-1}(1.4286) = 55^\circ$$

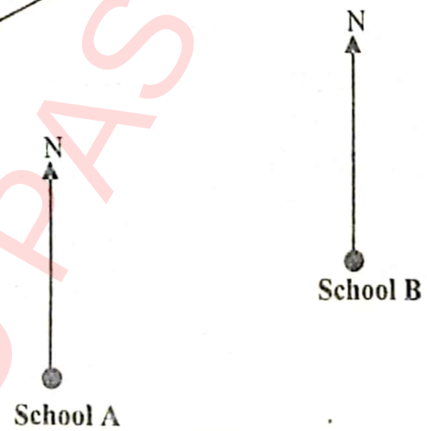
Thus, bearing of C from A = $040^\circ + 055^\circ = 095^\circ$

EXERCISE 6.6

1. Measure three figure bearing of B from A with the help of protractor in each of the following.



2. School A and school B are marked, find the bearing of school A from school B.



3. Two boats A and B are 5km apart and the bearing of boat B from boat A is 250°. Draw the diagram showing relative positions of both boats, then find the bearing of boat A from boat B.
4. The positions of three ships A, B and C are such that the bearing of B from A is 045° and ship C is to the east of A. If bearing of C from B is 180° and $m\overline{AB} = 10$km, what is the bearing of A from C? Also find the distance between ship A and ship C.
5. Babar Azam walked from point P to point Q at a bearing of 050° for 3km. Then, he walked from point Q to point R at a bearing of 140°. Find the distance QR, if the distance between P and R is 5km. Also find the bearing of R from P.
6. Rayyan spots a snake directly north from his location. Sarim is 25m east of Rayyan and spots the same snake. If the bearing of Sarim from snake is 122°, how far is Rayyan from the snake?
7. Bilal traveled 12km on his bike to reach school at a bearing of 060° from his home. Then, he went to football ground from school at a bearing of 150°. If the football ground is at a distance of 5km from school, find the distance he has to cover to reach home from football ground?
8. A car leaves the garage at a beaing of 040° and travels in a straight line for 13km. How many kilometres north and how many kilometres east has the car travelled from the garage?

KEY POINTS

- There are three systems of measurement of an angle.
  - i. Sexagesimal System or English System
  - ii. Radian measure or Circular System
  - iii. Centesimal or French System
- If a circle is divided into 360 equal arcs. The central angle of each arc is of measure 1 degree (1°).
- There are 60 minutes in a degree. Similarly, there are 60 seconds in a minute.
- To convert the measure of an angle from degrees to minutes or from minutes to seconds we multiply its value by 60.
- To convert measure of an angle from seconds to minutes or from minutes to degrees we divide its value by 60.
- Ratio between length of circular arc and radius is called radian.
- One radian is measure of central angle of an arc whose length is equal to radius of the circle.
- Measure of a complete angle in circular system is 2π radians.
- 1 radian $= \left(\frac{180}{\pi}\right)^0 \approx 57^0 17' 45''$ and $1^0 = \frac{\pi}{180}$ radians ≈ 0.0175 radian.
- Length of a circular arc is product of radius and radian measure of central angle. i.e. $l = r\,\theta$
- Two angles having same terminal ray in standard position are called coterminal angles.
- If terminal ray of an angle in standard position coincides with any axis, then it is called quadrantal angle. Measure of a quadrantal angle is multiple of 90^0.
- A circle whose radius is 1unit is called a unit circle.
- Pythagorean Identities:
  (i) $\sin^2\theta + \cos^2\theta = 1$ (ii) $\sec^2\theta - \tan\theta = 1$
  (iii) $\text{cosec}^2\theta - \cot^2\theta = 1$
- Bearings are the angles measured clockwise from north line and are written as three digit numbers.
- North is at the bearing of 000^0, east is at 090^0, south is at 180^0, west is 270^0.

MISCELLANEOUS EXERCISE 6

1. Encircle the correct option in each of the following.

(i) If $\sin x = \frac{1}{4}$, what is the value of $\cos x$?

(a) $\frac{\sqrt{15}}{4}$ (b) $\frac{3}{4}$ (c) $\frac{\sqrt{17}}{4}$ (d) $\frac{15}{4}$

(ii) $\cos\frac{2\pi}{3} = ?$

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$ (c) $-\sqrt{\frac{3}{2}}$ (d) $\sqrt{\frac{3}{2}}$

(iii) $\frac{1}{1-\sin\theta}+\frac{1}{1+\sin\theta}=?$

(a) $\operatorname{cosec}^2\theta$ (b) $2\operatorname{cosec}^2\theta$ (c) $\sec^2\theta$ (d) $2\sec^2\theta$

(iv) $-3-3\tan^2\theta=?$

(a) $3\operatorname{cosec}^2\theta$ (b) $-3\operatorname{cosec}^2\theta$ (c) $-3\sec^2\theta$ (d) $3\sec^2\theta$

(v) If $\cos x = \sin x$, then the value of x is

(a) 30° (b) 45° (c) 60° (d) 90°

(vi) $4\operatorname{cosec}^2\theta-4\cot^2\theta-4\cos 0^\circ=?$

(a) 0 (b) 1 (c) −1 (d) 4

(vii) If $\tan x = \frac{a}{b}$, then $\sin x = ?$

(a) $\frac{a}{\sqrt{a^2-b^2}}$ (b) $\frac{b}{\sqrt{a^2-b^2}}$ (c) $\frac{a}{\sqrt{a^2+b^2}}$ (d) $\frac{b}{\sqrt{a^2+b^2}}$

(viii) $50^\circ 30' = ?$

(a) 50.2° (b) 50.3° (c) 50.4° (d) 50.5°

(ix) $(1-\cos^2\theta)\sec^2\theta=?$

(a) $\operatorname{cosec}^2\theta$ (b) $\cot^2\theta$ (c) $\sec^2\theta$ (d) $\tan^2\theta$

(x) In which quadrant do the angles between 90° and 180° lie?

(a) 1st (b) 2nd (c) 3rd (d) 4th

(xi) The system in which the angles are measured in radians is called

(a) CGS (b) sexagesimal (c) circular (d) centesimal

(xii) $270^\circ = ?$

(a) $\frac{3\pi}{2}$ (b) $\frac{\pi}{2}$ (c) $-\frac{\pi}{2}$ (d) $-\frac{3\pi}{2}$

(xiii) What is the bearing of south-west direction line?

(a) 45° (b) 135° (c) 225 (d) 270°

(xiv) If an automobile travels at the bearing of is 070° from city A to city B , then the bearing from city B to city A is

(a) 070° (b) 110° (c) 250° (d) 290°

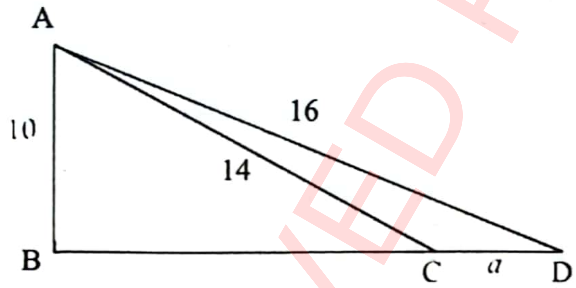
(xv) A boat sails 4km north from point L to M, then it sails 3km west from M to P. What is the bearing of P from L? Provide your answer to the nearest degree.

(a) 037° (b) 053° (c) 217° (d) 323°

2. Find $\sin\alpha$, $\tan\alpha$ and $\sec\alpha$, if $\cos\alpha = \frac{-5}{13}$, where $\frac{\pi}{2} < \alpha < \pi$.

3. The angle of elevation of a hot air balloon 500 m away, climbing up vertically, changes from 30° to 60° in 100 seconds. What is the upward speed in meters per second?

4. Find the value of a in the following figure.



5. The area of a right triangle is $50cm^2$. One of its angles is 45°. Find the lengths of the sides and hypotenuse of the triangle.

6. If the shadow of a building increases by 12 meters when the angle of elevation of the sun rays decreases from 60° to 45°, what is the height of that building?

7. From the top of a 100 meters high building, the angle of depression to the bottom of a second building is 30°. From the same point, the angle of elevation to the top of the second building is 20°. Calculate the height of the second building.

8. A plane is 119km west and 100km south of an airport. At what bearing, the pilot wants to fly directly back to the airport?

UNIT
07

COORDINATE GEOMETRY

In this unit the students will be able to:

- Use conventions for coordinates in the Cartesian plane for the derivation of distance formula.
- Represent and identify collinear and non collinear points.
- Find distance between two points in the plane.
- Derive mid-point formula and calculate mid-point of a line segment.
- Apply distance and mid-point formulas to solve real life situations.

Coordinate geometry plays a vital role in our daily life. The adjoining figure shows a lamp post which is equidistant from two homes, so that both the homes may get light from it equally. To erect the lamp post at this point, an idea from coordinate geometry is taken. Coordinate geometry helps us in analyzing characteristics and properties of two-dimensional geometrical shapes and develop mathematical arguments about geometrical relationships.

Distance Between Points on a Coordinate Line

We use the concept of absolute value to define the distance between given points on a coordinate line.

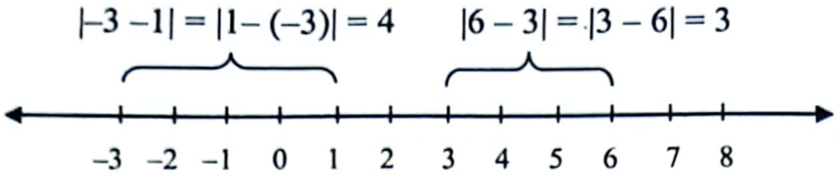


Fig. (i)

We note that the distance between the points with coordinates 3 and 6, shown in above fig.(i) equals 3 units. This distance is the difference obtained by subtracting the smaller (left most) coordinate from the larger (right most) coordinate $(6 - 3 = 3)$.

If we use absolute values, (since $|6 - 3| = |3 - 6| = 3$), then order of subtraction does not matter.

Let a and b be the coordinates of two points A and B respectively, on a coordinate line. The distance between A and B, denoted by d (A, B) is defined by

$$d(A, B) = AB = |b - a|$$

The number d (A , B) is the length of the line AB,

and $$d(A, B) = d(B, A)$$
$$|b - a| = |a - b|$$

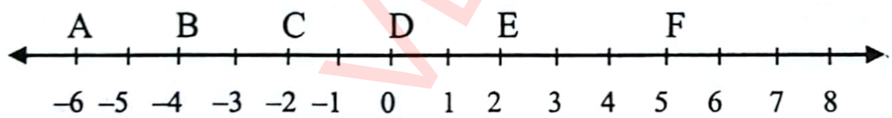
So, we have

$$AB = d(A, B) = d(B, A) = BA$$

Hence, the formula $d(A, B) = |b - a|$ is true regardless the sign of a and b.

Example 1:

A, B, C, D, E and F have coordinates – 6, – 4, – 2, 0, 2 and 5 respectively on a coordinate line as shown below.



Find (a) d (A , B) (b) d (A , C) (c) d(B , E)
(d) d (D, F) (e) d (A , F)

Solution: By definition we know that

(a) $d(A, B) = |b - a|$, a and b are the co-ordinates of points A and B respectively.

$a = -6$, $b = -4$

$d(A, B) = AB = |-4 - (-6)| = |-4 + 6| = 2$

Check Point

Can the distance be negative between any two points?

Similarly,

(b) $d(A, C) = AC = |-2 - (-6)| = |-2 + 6| = 4$

(c) $d(B, E) = BE = |2 - (-4)| = |2 + 4| = 6$

(d) $d(D, F) = DF = |5 - 0| = |5| = 5$

(e) $d(A, F) = AF = |5 - (-6)| = |6 + 5| = 11$

The Distance Formula

Given the two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, the distance between these points is given by the formula:

$$d(P_1, P_2) = P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Proof: If $x_1 \neq x_2$ and $y_1 \neq y_2$, then plot two points P_1 and P_2 with coordinates (x_1, y_1) and (x_2, y_2) respectively as illustrated in figure (ii).

Key Fact

$P_1P_2 = d(P_1, P_2)$
$P_1P_2^2 = (P_1P_2)^2$

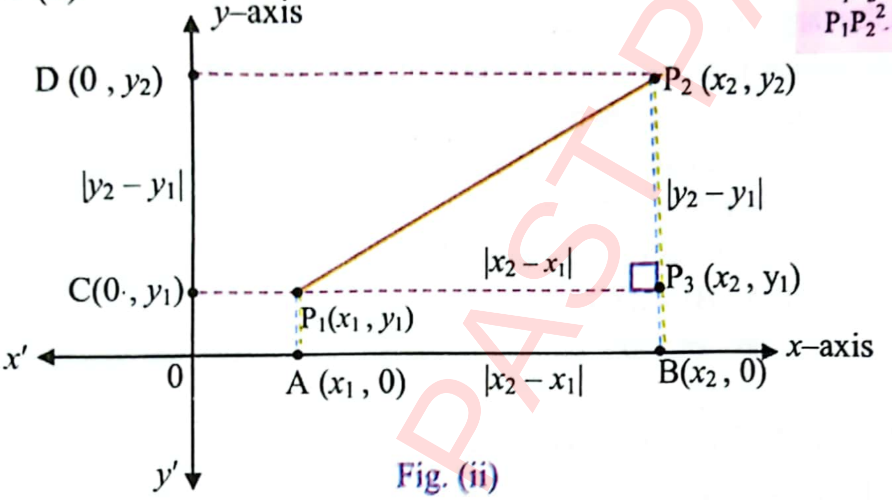


Fig. (ii)

In the above figure P_1A and P_2B are perpendiculars drawn from two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ on x–axis, and P_1C and P_2D are perpendiculars on y–axis.

The coordinates of the points A, B, C and D are also shown in figure accordingly.

Let the point where P_1C meets P_2B be P_3, with coordinate (x_2, y_1) so that $\Delta P_1P_2P_3$ is a right triangle with right angle at P_3.

From figure, we have

$d(A, B) = AB = P_1P_3 = |x_2 - x_1|$

and $d(C, D) = CD = d(P_2, P_3) = P_2P_3 = |y_2 - y_1|$

We next use the Pythagoras Theorem in right triangle $P_1P_2P_3$

$$[d(P_1, P_2)]^2 = [d(P_1, P_3)]^2 + [d(P_2, P_3)]^2$$

or

$$P_1P_2^2 = P_1P_3^2 + P_2P_3^2$$

$$P_1P_2^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$$

$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

So $d(P_1P_2) = P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

Key Fact

- In general $(x, y) \neq (y, x)$ because (x, y) is an ordered pair and order of the elements in an ordered pair cannot be changed.
- $(x, y) = (y, x)$ if and only if $x = y$.
- If $(x, y) = (2, 3)$, it means $x = 2$ and $y = 3$ and $(2,3) \neq (3,2)$.

Example 2:

Find the distance between the two points A (–1 , 3) and B (4 , 15).

Solution:

By distance formula distance between two points with coordinates (x_1 , y_1) and(x_2 , y_2) is :

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Let (–1 , 3) be denoted by (x_1 , y_1) i.e. $x_1 = -1$, $y_1, = 3$ and (4 ,15) be denoted by (x_2 , y_2) i.e. $x_2 = 4$, $y_2 = 15$

Now put them into the distance formula

$$AB = d = \sqrt{(4-(-1))^2+(15-3)^2}$$
$$d = \sqrt{(4+1)^2+(12)^2}$$
$$d = \sqrt{5^2+12^2}$$
$$d = \sqrt{25+144}$$
$$d = \sqrt{169}$$
$$d = 13$$

History a Mystery

Descartes was creative in many fields. But he is best known for his contribution to Mathematics. He had a habit of lying in bed for extended periods. One day, while watching a fly crawling on the ceiling, he set about trying to describe the path of the fly in mathematical language. Thus, the study of lines and curves was born.

Check Point

Find the distance between A $(\frac{1}{2}, -\frac{3}{2})$ and $(\frac{9}{2}, -\frac{8}{2})$

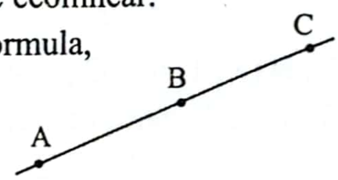
Collinear and Non-Collinear Point

A set of points that lie on a same straight line are said to be ccollinear.

If three points A, B and C are collinear then, by distance formula, we have:

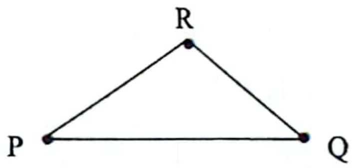
d (A , C) = d (A , B) + d (B , C)

or AC = AB + BC



When three or more points do not lie on the same straight line, they are said to be non–collinear points.

In the figure, points P , Q and R are non-collinear



Application of Distance Formula

Example 3:

Prove that A (3 , 2), B (4 , 4) and C (5 , 6) are collinear points.

Solution:

In first step we find the distance between them.

Memory Plus

Try to find the coordinates of
a) Islamabad
b) Pakistan
c) your native village

By **Distance Formula,** distance between two points $(x_1\ ,\ y_1)$ and $(x_2\ ,\ y_2)$ is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$d\,(A\ ,\ B) = AB = \sqrt{(4-3)^2 + (4-2)^2}$$

$$AB = \sqrt{1^2 + 2^2} = \sqrt{1+4}$$

$$AB = \sqrt{5} \quad \text{......... (i)}$$

$$d\,(B\ ,\ C) = BC = \sqrt{(5-4)^2 + (6-4)^2}$$

$$BC = \sqrt{1^2 + 2^2} = \sqrt{1+4}$$

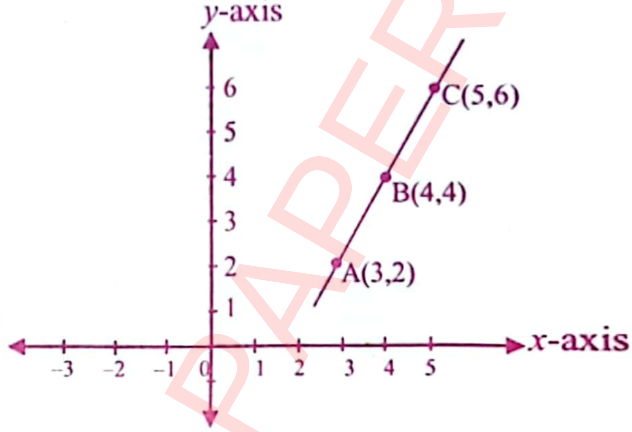
$$BC = \sqrt{5} \quad \text{........ (ii)}$$

$$d\,(A\ ,\ C) = AC = \sqrt{(5-3)^2 + (6-2)^2}$$

$$AC = \sqrt{2^2 + 4^2}$$

$$= \sqrt{4+16} = \sqrt{20}$$

$$= 2\sqrt{5} \quad \text{........ (iii)}$$



From (i), (ii) and (iii), we find that

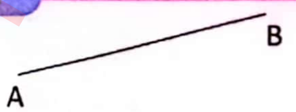
$$AC = AB + BC$$

$$2\sqrt{5} = \sqrt{5} + \sqrt{5}$$

$$2\sqrt{5} = 2\sqrt{5}$$

Hence, the given points are collinear.

Key Fact



(i) Line segment AB is denoted by $\overline{AB}$

(ii) Length of $\overline{AB}$ is denoted by AB or $m\overline{AB}$ or $|\overline{AB}|$.

(iii) Distance between points A and B is denoted by AB or d(A, B) or $m\overline{AB}$ or $|\overline{AB}|$.

Example 4:

Prove that the points (3 , 4), (3 , 1) and (8 , 4) are the vertices of a right triangle.

Solution:

Suppose A (3 , 4), B (3 , 1) and C (8 , 4) are the vertices of a triangle.

In first step we find the distance between them.

$$d\,(A\ ,\ B) = AB = \sqrt{(1-4)^2 + (3-3)^2}$$

$$= \sqrt{(-3)^2 + 0^2}$$

$$AB = \sqrt{9+0}$$

$$AB = 3$$

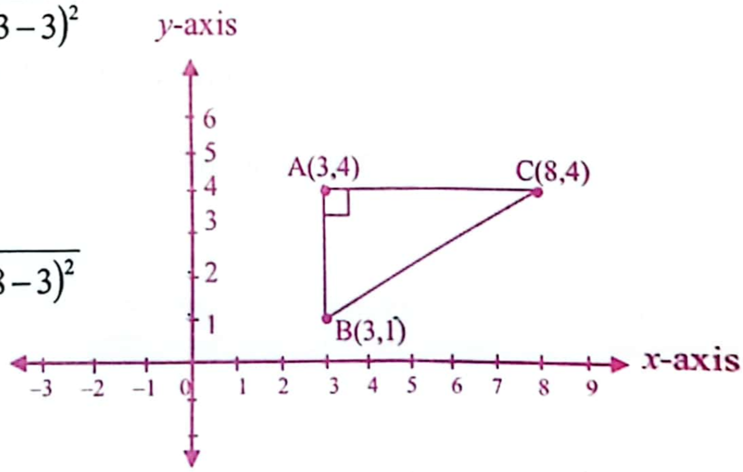
$$d\,(B\ ,\ C) = BC = \sqrt{(4-1)^2 + (8-3)^2}$$

$$= \sqrt{3^2 + 5^2}$$

$$= \sqrt{9+25}$$

$$BC = \sqrt{34}$$

$$d\,(A\ ,\ C) = AC = \sqrt{(8-3)^2 + (4-4)^2}$$



$$AC = \sqrt{25+0} = 5$$

In right triangle we know that the greatest length is always the hypotenuse. So, BC is the hypotenuse. By Pythagoras Theorem,

$$BC^2 = AB^2 + AC^2$$
$$\left(\sqrt{34}\right)^2 = 5^2 + 3^2$$
$$34 = 25 + 9$$
$$34 = 34$$

Hence; A , B and C form a right triangle.

Check Point

Prove by distance formula that the points (–2 , 0), (4 , 0), (4 , 2) and (–2 , 2) are the vertices of a rectangle.

EXERCISE 7.1

1. Find the distance between each pair of points.

(i) (– 2, 1) and (1, 5)
(ii) (1 , 1) and (7 , 9)
(iii) (0, – 5) and (– 2, 1)
(iv) (0 , – 5) and (10 , 5)
(v) (4, –1) and (2, 0)
(vi) (– 3 , 1) and (– 2 , – 3)
(vii) $\left(\frac{1}{2}, 2\right)$ and (– 2, 1)
(viii) $\left(\frac{7}{2}, 11\right)$ and $\left(13, \frac{17}{2}\right)$
(ix) $\left(-\frac{9}{4}, -\frac{7}{3}\right)$ and $\left(-\frac{3}{2}, -\frac{5}{4}\right)$
(x) $\left(2\frac{1}{2}, 5\frac{1}{4}\right)$ and (5, –1)

2. Check by distance formula whether these points are collinear or not.

(i) (0 , 1), (2 , 3) and (3, 4)
(ii) (– 5, – 4), (1, 0) and (6, 7)
(iii) (0, – 5), (3, 7) and (5, 15)
(iv) (6, 3), (14, 7) and (– 6, – 3)
(v) (0, 15), (2, 9), and (7, – 6)
(vi) (1, – 2), (7, 8) and (– 2, – 7)

3. Check whether the following points are the vertices of

(a) an equilateral triangle,
(b) an isosceles triangle,
(c) a right triangle,
(d) a scalene triangle.

(i) (2, 3), (5 , 3) and (2 , 1)
(ii) (0 , 0), (1 , $\sqrt{3}$) and (2 , 0)
(iii) (0 , 1), (8 , 4) and (0 , 8)
(iv) (– 3 , – 5), (3 , – 3) and (0 , 6)
(v) (–1 , –1) , (–1 , 5) and (4 , 2)
(vi) (1 , 2), (7 , 2) and (7 , 8)

Also find the perimeter of triangle in each case.

4. Show that A (– 4 , 2), B (1 , 4) , C (3 , – 1) and D (– 2 , – 3) are the vertices of a square.
5. Show that A (– 4 , – 1), B (0 , –2), C (6 , 1) and D (2 , 2) are the vertices of a parallelogram.
6. Show that A (1 , 2) , B (7 , 2) , C (7 , 8) and D (1 , 8) are the vertices of a rectangle. Prove that its diagonals are also equal in length.
7. Show that A (4 , 2), B (7 , 2), C (7 , 5) and D (4 , 5) are the vertices of a square. Also show that the length of its diagonals is same.
8. Are A (– 6 , 2), B (1 , 2), C (1 , 5) and D (– 6 , 5) the vertices of a rectangle? Also plot them.
9. Prove that the points A (–3, 0), B (3, 0), C (6, 4) and D (0, 4) are the vertices of a parallelogram.
10. Show that A (2 , –1), B (8 , –1), C (8 , 3) and D (2 , 3) are the vertices of a rectangle and prove that Δ ABC and Δ ABD form right triangles.

Mid-Point of Two given Points in a Coordinate Plane

Suppose, we have two points (2 , 4) and (6 , 10) and we want to find the distance between them. First, we plot the points on the graph and join them. Then we can draw the lines that form a right triangle by using these points as two of the corners.

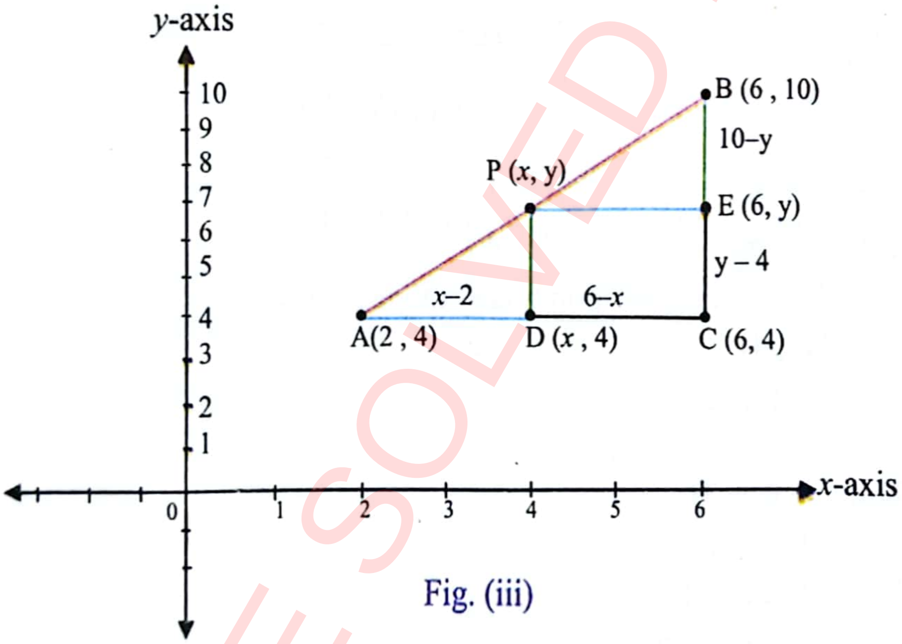


Fig. (iii)

In the diagram, Δ ABC is drawn by drawing AC parallel to the x–axis and BC parallel to the y– axis. The coordinates of C are (6 , 4), where two lines AC and BC intersect.

Let P (x , y) be the mid point of AB. Draw PD parallel to the y–axis and PE parallel to the x–axis. Then the coordinate of D and E are (x , 4) and (6 , y) and, ΔAPD and ΔPBE are congruent by (ASA).

We have AD = PE , PD = BE

$$x - 2 = 6 - x \quad , \quad y - 4 = 10 - y$$
$$2x = 8 \quad , \quad 2y = 14$$
$$x = 4 \quad , \quad y = 7$$

The mid point of AB is P (4 , 7).

We can also find the mid point of A (2 , 4) and B (6 , 10) in this way

$$P\left(\frac{2+6}{2}, \frac{4+10}{2}\right)$$
$$= P\,(4\,,7)$$

Mid-Point Formula

In the above figure (iii) if we take A (x_1 , y_1) instead of A (2 , 4) and B (x_2 , y_2) instead of B (6 , 10), then we have:

$$x - 2 = x - x_1 \text{ and } \quad 6 - x = x_2 - x$$

Similarly, $$y - 4 = y - y_1 \text{ and } \quad 10 - y = y_2 - y$$

We have already proved that

$$AD = PE \quad , \quad PD = BE$$
$$x - x_1 = x_2 - x \quad , \quad y - y_1 = y_2 - y$$
$$2x = x_1 + x_2 \quad , \quad 2y = y_1 + y_2$$
$$x = \frac{x_1 + x_2}{2} \quad , \quad y = \frac{y_1 + y_2}{2}$$

To apply the mid point formula **remember** that:

the x–coordinate of the midpoint = the average of the x–coordinates

and the y–coordinate of the midpoint = the average of the y–coordinates.

The mid-point P of the line segment from $P_1\,(x_1, y_1)$ to $P_2\,(x_2, y_2)$ is

$$P\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

Example 5:

By using mid point formula, prove that the diagonals of a parallelogram with vertices (1 , 1), (2 , 4), (6 , 1) and (7 , 4) bisect each other.

Solution:

First we plot these vertices on a Cartesian plane as shown in the fig.(iv). In parallelogram ABCD, AC and BD are its diagonals.

Let both the diagonals intersect each other at point $E(x , y)$.

By mid point formula, mid point of AC has the coordinates as

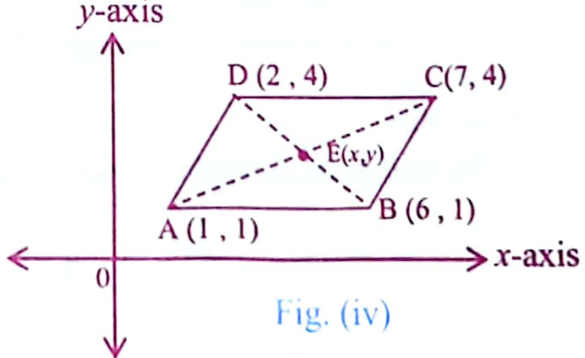


Fig. (iv)

$$\left(\frac{7+1}{2},\frac{4+1}{2}\right)$$

$$=(4\ ,2.5)\ \ldots\ldots\ (\text{i})$$

and the coordinates of the mid point of diagonal BD is $\left(\frac{2+6}{2},\frac{4+1}{2}\right)$

$$=(4\ ,2.5)\ \ldots\ldots\ (\text{ii})$$

From (i) and (ii) both the diagonals have the same mid point, so they bisect each other at point E(4 , 2.5).

Example 6:

If (1 , 4) is the midpoint of the line segment joining (6 , b) and (a , 2). Find a and b.

Solution:

By mid point formula, mid point of the line segment with end points (x_1 , y_1) and (x_2 , y_2) is

$$\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right).$$

So, mid point of the line segment joining the end points (6 , b) and (a , 2) is

$$\left(\frac{6+a}{2},\frac{b+2}{2}\right).$$

By using the given condition, we have

$$\left(\frac{6+a}{2},\frac{b+2}{2}\right)=(1,4)$$

By the definition of equal ordered pairs,

$$\frac{6+a}{2}=1 \quad \text{and} \quad \frac{b+2}{2}=4$$

$$6+a=2 \quad \text{and} \quad b+2=8$$

$$a=2-6 \quad \text{and} \quad b=8-2$$

$$a=-4 \quad \text{and} \quad b=6$$

Hence, $a=-4$ and $b=6$

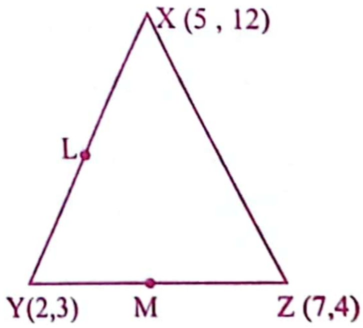
Example 7:

Use ΔXYZ in the adjoining figure to find the following.

(i) Midpoint L and M of XY and YZ respectively

(ii) d (L , M) and d(X , Z)

(iii) The relationship between LM and XZ .



Solution:

(i) By mid point formula the coordinates of L

are $\left(\frac{2+5}{2},\frac{12+3}{2}\right)=\left(\frac{7}{2},\frac{15}{2}\right)$

Coordinates of M are $\left(\frac{2+7}{2}, \frac{3+4}{2}\right) = \left(\frac{9}{2}, \frac{7}{2}\right)$

(ii) $d(L, M) = \sqrt{\left(\frac{7}{2} - \frac{9}{2}\right)^2 + \left(\frac{15}{2} - \frac{7}{2}\right)^2}$

$= \sqrt{\left(\frac{-2}{2}\right)^2 + \left(\frac{8}{2}\right)^2} = \sqrt{1+16} = \sqrt{17}$

Now, $d(X, Z) = \sqrt{(12-4)^2 + (5-7)^2}$

$= \sqrt{8^2 + (-2)^2}$

$= \sqrt{64+4} = \sqrt{68} = 2\sqrt{17}$

(iii) From part (ii), we observe that

$d(X, Z) = 2$ times $d(L, M)$

Or $XZ = 2\,LM$

Or $LM = \frac{XZ}{2}$

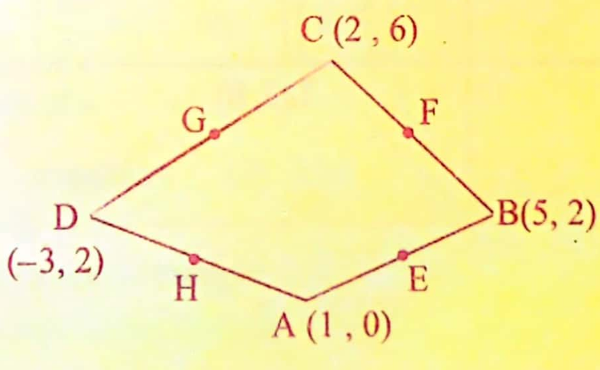
Hence,

Length of the line which is obtained by joining the mid points of any two sides of a triangle is half of the length of third side.

EXERCISE 7.2

1. Find the mid point of the line segment joining the given points.
   (i) (2 , 5) , (6 , 9) (ii) (–1 , 0) , (2 , 2)
   (iii) (1.4 , –1.5) , (2.6 , 3.5)

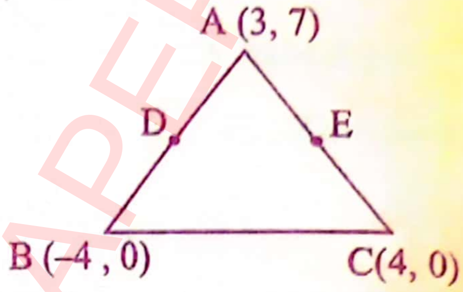
2. Look at the quadrilateral ABCD in the adjoining figure If E, F, G and H are the mid points of its sides as shown in the figure, then
   (i) Find the coordinates of E, F, G and H
   (ii) Draw line segments EF, FG, GH and HE by joining the mid points of its sides.
   (iii) Find the distance of EF, FG, GH and HE
   (iv) Guess the type of quadrilateral EFGH.



3. Check by using the mid point formula whether the diagonals of trapezium PQRS with vertices at P(1 , 0) , Q(6 , 0) , R(7 , 4) , S(−1 , 4) bisect each other or not.

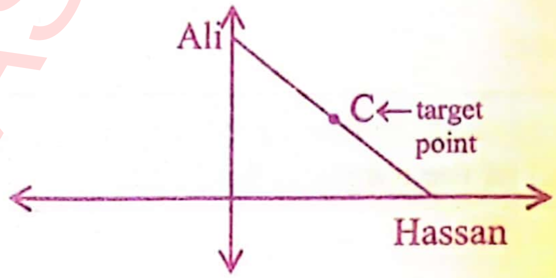
4. If vertices of a rhombus are A (−3,−1), B (0, 0), C (1, 3) and D (−2, 2) Show that its diagonals bisect each other. Also find the length of each diagonal.

5. Look at the Δ ABC in the adjoining figure.



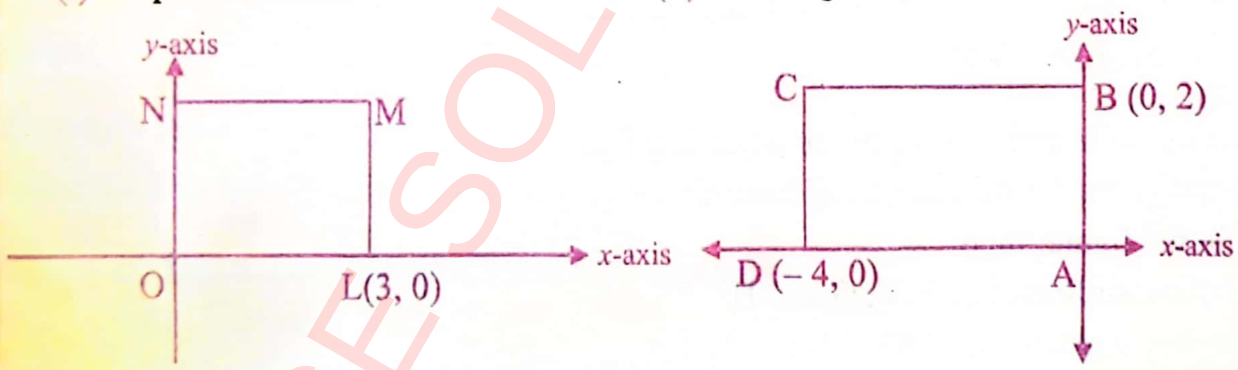
   (i) Find the coordinates of the mid points D and E of AB and AC respectively.
   (ii) Find d(D , E) and d(B , C).
   (iii) Find half of d (B , C).
   (iv) Compare the length of DE and BC and draw conclusion.

6. Hassan and Ali participated in a race competition. They both have to start running from different points on coordinate axes at a distance of 6 units from origin but they must reach at the same target point C, after covering an equal distance. If Hassan starting point is on *x*-axis and Ali's starting point is on *y*-axis as shown in the figure, then find the coordinates of the:



   (i) starting points of Hassan and Ali (ii) target point C

7. Label the missing coordinates of the vertices of the following figures.
   (i) Square OLMN (ii) Rectangle ABCD



KEY POINTS

- The distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by
  $d(A, B) = AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
- Distance formula is used for finding the
  (a) distance between any two points on a line,
  (b) length of the sides of geometrical figures obtained by joining three or more non-collinear points. e.g. triangles and quadrilaterals (square, rectangle, parallelogram, rhombus, trapezium, kite etc.).
- All those points, which lie on a same straight line, are called collinear points.
- The points which do not lie on a same straight line are called non-collinear points. e.g. vertices of a triangle and of a quadrilateral are non-collinear points.
- The mid-point of a line segment with ends $A(x_1, y_1)$ and $B(x_2, y_2)$ is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$
- The mid point of the diagonals of a quadrilateral can be found by using the mid point formula.
- If two lines bisect each other then, their mid point will be same.

MISCELLANEOUS EXERCISE 7

. Encircle the correct option in the following.

(i) If the coordinates of four points are $A\left(\frac{3}{2}, -\frac{7}{4}\right)$, $B\left(3, \frac{3}{4}\right)$, $C(3, -2)$ and $D\left(1, \frac{1}{2}\right)$, the relationship between the distance AB and CD is:

(a) $AB > CD$ (b) $AB < CD$ (c) $AB = CD$ (d) $AB = \frac{1}{2}CD$

(ii) If a and b are any two points on a coordinate line (number line), then its mid point will be:

(a) $\frac{a+b}{2}$ (b) $\left(\frac{a}{2}, \frac{b}{2}\right)$ (c) $\left(\frac{a}{2}, b\right)$ (d) $\left(a, \frac{b}{2}\right)$

(iii) (5, 2), (7, 0), (1, – 2) and (–1, 0) are the vertices of a

(a) parallelogram (b) square (c) trapezium (d) kite

(iv) The triangle with vertices (1, –3), (3, 2) and (–2, 4) is:

(a) a right triangle (b) an isosceles triangle
(c) a right isosceles triangle (d) an equilateral triangle

(v) Mid-point of $(c, -d)$ and $(-d, c)$ is:

(a) $\left(\frac{c}{2}, -\frac{d}{2}\right)$ (b) $\left(\frac{-c-d}{2}, \frac{-d-c}{2}\right)$

(c) $\left(\frac{-d+c}{2}, \frac{c-d}{2}\right)$ (d) $(c-d, -d+c)$

(vi) If P(2, 3) is the mid point of A $(x, 4)$ and B $(-3, y)$, then the coordinates of (x, y) are:

(a) $\left(\frac{-1}{2}, \frac{3+b}{2}\right)$ (b) $(5, -1)$ (c) $\left(\frac{x-3}{2}, \frac{y+4}{2}\right)$ (d) (7, 2)

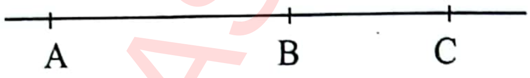
(vii) If two ordered pairs (4, 5) and $\left(\frac{a+1}{2}, b-3\right)$, are equal then, the value of a is:

(a) 4 (b) 8 (c) 7 (d) 5

(viii) In which quadrant, the point $(-4, -(-6))$ lies?

(a) first (b) second (c) third (d) fourth

(ix) In the adjoining figure of a line:



d (A , C) – d (B , C) =

(a) d (B , A) (b) d (C , B) (c) d (A , B) (d) both a & c

(x) The abscissa on y–axis is:

(a) 1 (b) x (c) zero (d) y

(xi) Minimum number of sides in a polygon is:

(a) 2 (b) 3 (c) 4 (d) 5

(xii) If $x < 0$ and $y > 0$, then $(-x, y)$ will lie in quadrant:

(a) I (b) II (c) III (d) IV

(xiii) The point (3, 5) is at minimum distance from:

(a) x–axis (b) y–axis (c) origin (d) both a & b

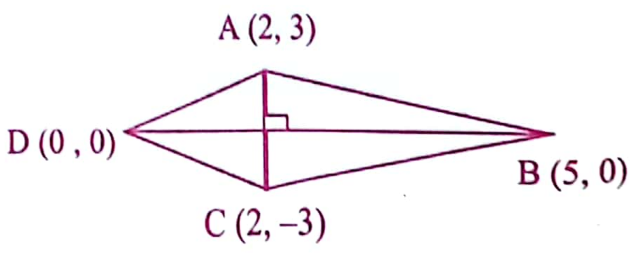
(xiv) If M is the mid point of the line segment LN, then LN : MN is:

(a) 2 : 1 (b) 1 : 2 (c) 1 : 1 (d) 1 : 3

(xv) All the bisectors of a line segment always pass through:

(a) x–axis (b) origin (c) mid-point (d) y–axis

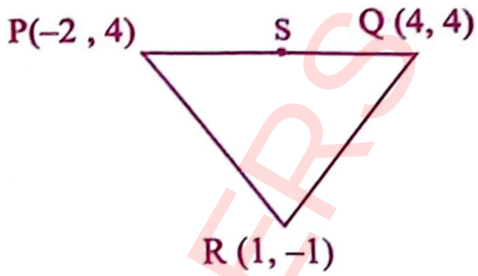
(xvi) The below figure is a kite, what will be the coordinates of the point of intersection of both diagonals?



(a) (0, 2) (b) $\left(2\frac{1}{2}, 0\right)$

(c) (0, 0) (d) (2, 0)

(xvii) In the given figure, if S is the mid point of PQ, then SR is



(a) $\sqrt{2}$ (b) ± 5
(c) 6 (d) 5

(xviii) If distance between two points A(c, 10) and B (0, –1) is 20 units, then the value of c is:
(a) 279 (b) $3\sqrt{31}$ (c) 3 (d) 40

xix) The length of the hypotenuse of ΔXYZ whose vertices are (7, 4), (7, 1) and (–3, 1) is:
(a) 3 (b) 10 (c) $\sqrt{109}$ (d) $\sqrt{91}$

xx) The mid point of the hypotenuse of a right triangle with vertices (4, 0), (2, 1) and (–1, – 5) is:

(a) $\left(2, \frac{1}{2}\right)$ (b) $\left(\frac{1}{2}, -2\right)$ (c) $\left(\frac{3}{2}, -\frac{5}{2}\right)$ (d) $\left(\frac{3}{2}, \frac{5}{2}\right)$

xxi) If end points of a diameter of the circle are (4, 5) and (6, 9), then coordinates of its center will be:
(a) (7, 5) (b) (1, 2) (c) (5, 7) (d) (0, 0)

xxii) If two points (2, 3) and (4, 5) are at equal distance from (3, t) then, the value of t is:
(a) 5 (b) 4 (c) 3 (d) 7

xxiii) A point P(x, y) is at equal distance from both the axes and lies in the third quadrant. If it is at a distance of 4 units from y–axis, then its coordinates are:
(a) (– 4, 4) (b) (4, – 4) (c) (4, 4) (d) (– 4, – 4)

xiv) A line AB intersects x–axis at C with a distance of 3 units from origin and y–axis at D with a distance of 6 units from origin, then d (C, D) is:
(a) $3\sqrt{5}$ (b) $3\sqrt{3}$ (c) 3 (d) 45

The consecutive vertices of a quadrilateral are A (0, 0), B (0, 5), C(4, 7) and D(4, 2). Show that quadrilateral is a parallelogram.

Show that the four points (5, 8), (7, 5), (3, 5) and (5, 2) are the vertices of a rhombus.

A (3, 4), B(–1, 7) and C (x, y) are three collinear points such that B is midpoint of $\overline{AC}$. Find values of x and y.

UNIT
08

GEOMETRY OF STRAIGHT LINES

In this unit the students will be able to:

- Find the gradient of straight line when coordinates of two points in a plane are given.
- Find the gradient of parallel and perpendicular lines.
- Derive equations of straight line in slope-intercept form, point-slope form, two points form, intercepts-form, symmetric form and normal form.
- Show that the linear equation in two variables represents a straight line.
- Reduce the general form of the equation of a straight line to other standard forms.
- Find the angle between two coplanar intersecting lines.
- Find the equation of family of lines passing through the point of intersection of two given lines.
- Calculate angles of triangles when the slopes of sides are given.
- Apply concepts of coordinate geometry to real world problems.

To go from one floor to another at a library, you can take either the stairs or the escalator. You can climb stairs at a rate of 1.75 feet per second, and the escalator rises at a rate of 2 feet per second. You have to travel vertical distance of 28 feet. The equations showing the vertical distance d (in feet) traveled after t seconds are:

Stairs: $d=-1.75t+28$ **Escalator:** $d=-2t+28$

a. Graph the equations in the same coordinate plane.
b. How much time do you save by taking the escalator?

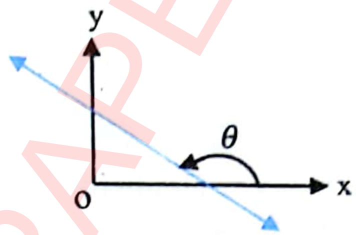
Gradient (slope) of Straight Line

Inclination of Straight Line

The inclination of a straight line is the angle θ which the line makes with positive x-axis measured in anti-clockwise direction.

Key Fact

- Any line parallel to x-axis is called horizontal line.
- Any line parallel to y-axis is called vertical line.
- Line which is neither horizontal nor vertical is called oblique line.
- Inclination of horizontal line is $0°$.
- Inclination of vertical line is $90°$.



Gradient

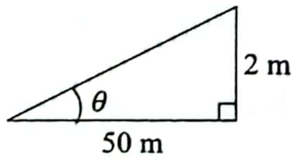
When a car rides up a hill, its speed reduces. The steeper the hill, the harder it is to climb.

The measure of the steepness of hill is called gradient (slope).

Gradient is ratio of the vertical distance to the horizontal distance.

$$\text{Gradient} = \frac{\text{vertical distance}}{\text{horizontal distance}} = \frac{\text{rise}}{\text{run}}$$

$$= \frac{2}{50} = \frac{1}{25} = \tan\theta$$



Where the angle θ is inclination.

Key Fact

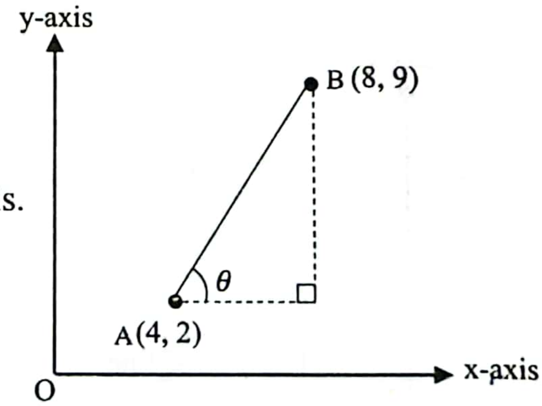
The gradient of a straight line is the tangent of the angle which the line makes with the positive direction of the x-axis.

Gradient of Straight Line

The gradient of straight line is defined as:

$$\tan\theta = \frac{\text{rise}}{\text{run}}$$

In the figure line AB is inclined at an angle θ with x-axis.



$$\text{Gradient of line AB} = \frac{\text{rise}}{\text{run}}$$

$$= \frac{\text{difference in y-coordinates of AB}}{\text{difference in x-coordinates of AB}}$$

$$= \frac{9-2}{8-4} = \frac{7}{4}$$

$$\text{Alternatively, gradient of AB} = \frac{2-9}{4-8} = \frac{-7}{-4} = \frac{7}{4}$$

Generally, gradient of a straight line is denoted by m.
Therefore, $m = \tan\theta$

Example 1:

(i) Find the gradient (slope) of line whose inclination is 45°.

(ii) Find inclination of line whose gradient is 1.732.

Solution:

(i) Here $\theta = 45^\circ$

$\therefore$ Gradient $= \tan 45^\circ = 1$

(ii) Gradient $= \tan\theta = 1.732$

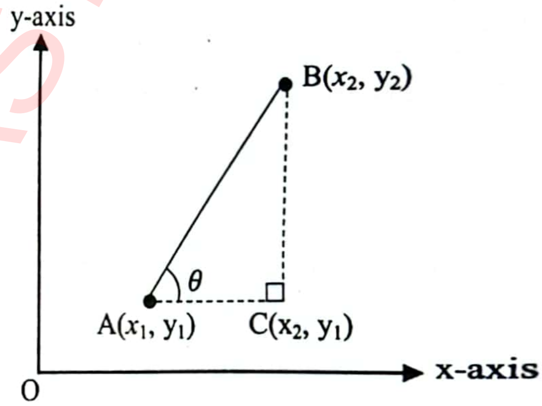
$\therefore$ Inclination $= \theta = \tan^{-1}(1.732) = 60^\circ$

Formula of Gradient

If $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points on a line AB, then gradient of AB is:

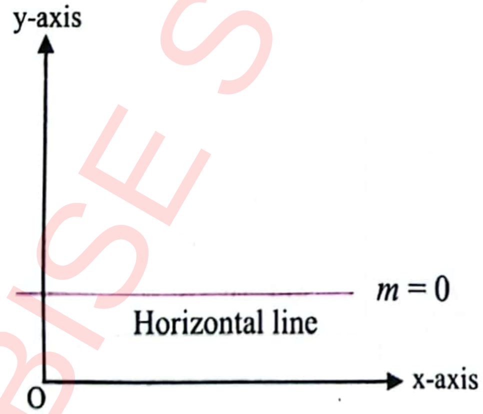
$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$$

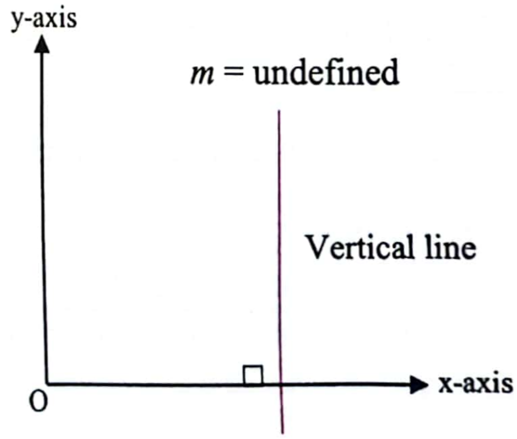
Gradient of AB can not be $\frac{y_1 - y_2}{x_2 - x_1}$ or $\frac{y_2 - y_1}{x_1 - x_2}$



Key Fact

- If the line is parallel to x-axis, then $\theta = 0^\circ$, and we have $m = \tan 0^\circ = 0$. Thus, gradient of horizontal line is 0.
- If the line is perpendicular to x-axis, then $\theta = 90^\circ$, and we have $m = \tan 90^\circ$ which is undefined. Thus, gradient of vertical line is undefined.





Sign of Gradient

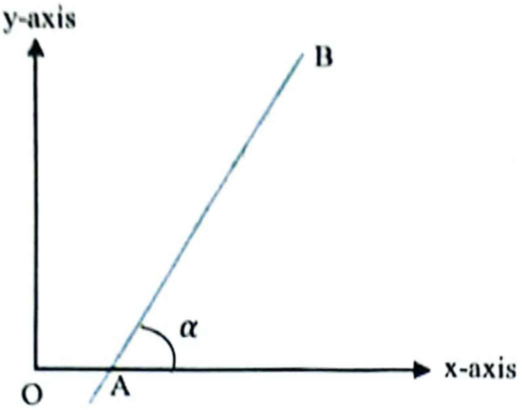


Figure 1

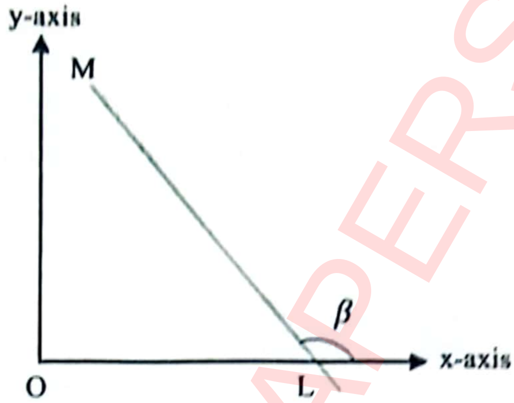


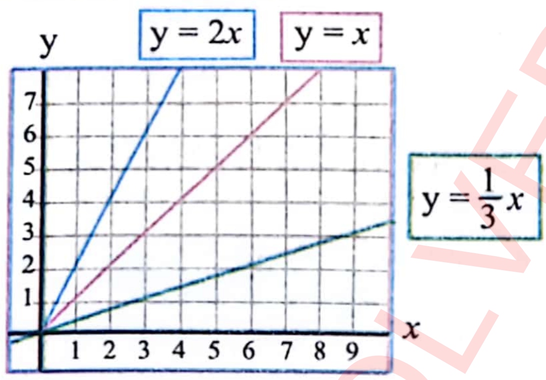
Figure 2

- In the figure 1, line AB makes an angle α $(0^\circ < \alpha < 90^\circ)$ with the positive x-axis. As the value of tan α is positive in first quadrant, therefore gradient of line AB is positive for $0^\circ < \alpha < 90^\circ$.
- In the figure 2, line LM makes an angle β $(90^\circ < \beta < 180^\circ)$ with the positive x-axis. As the value of tan β is negative in second quadrant, therefore gradient of line LM is negative for $90^\circ < \beta < 180^\circ$.

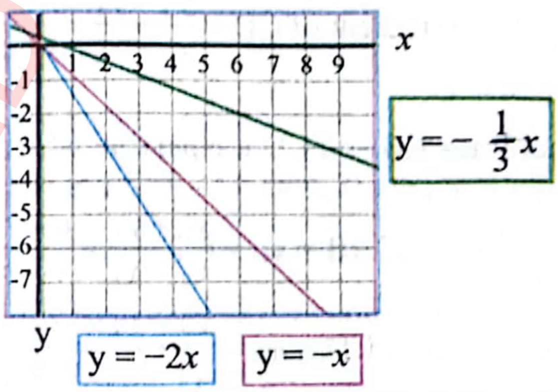
Key Fact

Gradients can be positive or negative but are always observed from left to right.

Positive Gradients



Negative Gradients



Gradient of Parallel and Perpendicular Lines

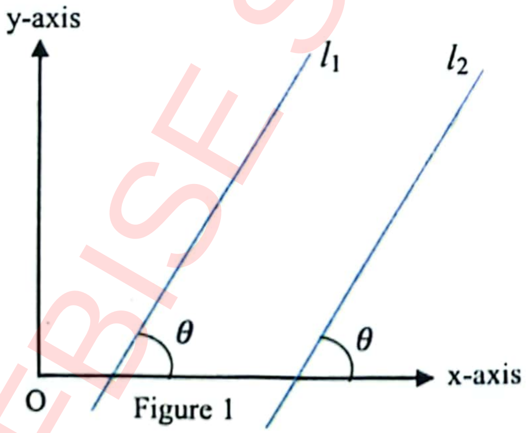


Figure 1

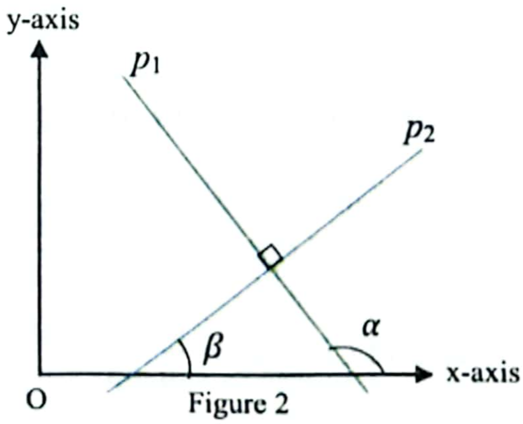


Figure 2

Figure 1 shows two parallel lines l_1 and l_2 making angle α with positive x-axis. If gradients of l_1 and l_2 are m_1 and m_2 respectively then:

$$m_1 = m_2$$

Figure 2 shows two perpendicular lines p_1 and p_2 making angles α and β with positive x-axis respectively. If gradients of p_1 and p_2 are m_1 and m_2 respectively then:

$$m_1 = -\frac{1}{m_2} \text{ or } m_2 = -\frac{1}{m_1}$$

Key Fact

- If two non-vertical lines are parallel, there gradients are equal.
- If two lines have same gradients, they are parallel.
- If m_1 and m_2 are gradients of two perpendicular lines, then:

$$m_1 \times m_2 = -1$$

- Three points A, B and C are collinear if gradients of AB, BC and AC are equal.

Example 2:

Find gradients of lines joining: (i) A(1, 4), B(3, 6) (ii) C(–3, 5), D(0, 3)

Solution:

(i) Gradient of AB $= \frac{y_2 - y_1}{x_2 - x_1} = \frac{6-4}{3-1} = 1$

(ii) Gradient of CD $= \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-5}{0-(-3)} = \frac{-2}{3}$

Example 3:

Prove that lines AB and CD are parallel where A(3, 5), B(4, 6), C(7, 11), D(9, 13).

Solution:

Gradient of AB $= m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6-5}{4-3} = 1$

Gradient of CD $= m_2 = \frac{13-11}{9-7} = \frac{2}{2} = 1$

As, $m_1 = m_2$

Therefore, AB and CD are parallel lines.

Check Point

Slope of line PQ is 5. Find the slope of line perpendicular to PQ.

Example 4:

Lines AB and CD with coordinates A(0, 3), B(2, 8), C(3, 10), D(8, k) are perpendicular. Find k.

Solution:

Gradient of AB $= m_1 = \frac{8-3}{2-0} = \frac{5}{2}$

Gradient of CD $= m_2 = \frac{k-10}{8-3} = \frac{k-10}{5}$

As, AB and CD are perpendicular,

Therefore, $m_1 \times m_2 = -1$

$\Rightarrow \quad \frac{5}{2} \times \frac{k-10}{5} = -1$

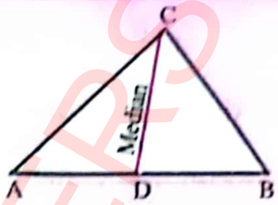
$\Rightarrow \quad \frac{5k-50}{10} = -1$

$\Rightarrow \quad 5k - 50 = -10 \quad \Rightarrow \quad 5k = -10 + 50$

$\Rightarrow \quad 5k = 40 \quad \Rightarrow \quad k = 8$

Key Fact

Line segments joining mid point of any side of a triangle with opposite vertex is called median of triangle.



EXERCISE 8.1

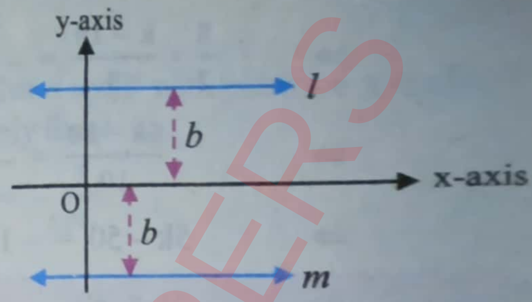
1. Find the gradient (slope) of line whose inclination is:
   (i) 0° (ii) 30° (iii) 60° (iv) 90°
   (v) 120° (vi) 150° (vii) 170° (viii) 45.5°
2. Find inclination of the line whose slope is:
   (i) 0 (ii) 0.577 (iii) – 1.732 (iv) – 0.364
3. Find gradient and inclination of lines joining:
   (i) A(2, 6), B(5, 8) (ii) C(–2, 4), D(1, –3) (iii) E(5, –2), F(–2, –3)
4. If A(–2, 6) and B(7, –3), find the slope of line:
   (i) parallel to AB. (ii) perpendicular to AB.
5. Find x if the slope of line passing through A(3, x), B(5, 8) is 4.
6. Find k if lines passing through A(k, 2), B(3, 5) and C(5, –1), D(8, 7) are parallel.
7. Find k if lines passing through P(–1, 2), Q(4, 7) and R(2, k), S(7, 10) are perpendicular.
8. Using slopes, prove that points X(0, –3), Y(4, 7) and Z(6, 12) are collinear.
9. Find value of y if points P(4, y), Q(5, 2) and R(6, 2y + 1) are collinear.
10. Prove by using slopes that points A(3, –1), B(–5, –5) and C(1, 3) are vertices of a right angled triangle.
11. Using slope, prove that A(–2, 1), B(6, 3), C(10, 5) and D(2, 3) are vertices of parallelogram.
12. P(x, y), Q(–2, 2), R(1, 4) and S(10, 1) are vertices of a parallelogram. Find P(x, y).
13. Three vertices of a rhombus are A(2, –1), B(3, 4) and C(–2, 3). Find fourth vertex.
14. If (5, 0), (0, 5) and (8, 8) are vertices of a triangle, find:
    (i) slopes of sides. (ii) slopes of medians. (iii) slopes of altitudes.

Equation of Straight Line Parallel to X-axis

In the figure, all the points on the line parallel to x-axis remain at a constant distance b from x-axis.

Therefore, all points on the line satisfy the equation:

$$y = b$$



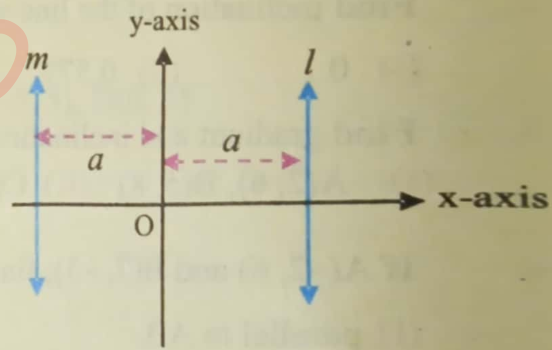
- If $b > 0$, then the line is above the x-axis (line l).
- If $b < 0$, then the line is below the x-axis (line m).
- If $b = 0$, then the line becomes x-axis and equation of x-axis become $y = 0$.

Equation of Straight Line Parallel to Y-axis

In the figure, all the points on the line parallel to y-axis remain at a constant distance a from y-axis.

Therefore, all points on the line satisfy the equation:

$$x = a$$

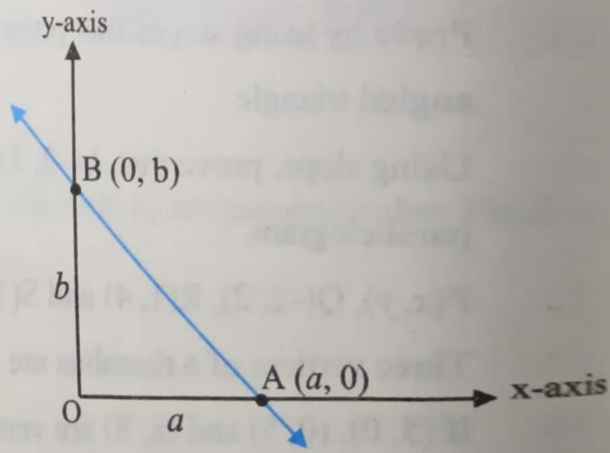


- If $a > 0$, then the line is on the right of the y-axis (line l).
- If $a < 0$, then the line is on the left of the y-axis (line m).
- If $a = 0$, then the line becomes y-axis and equation of y-axis become $x = 0$.

Intercepts

If a line AB intersects x-axis at $(a, 0)$, then a is called x-intercept of the line AB.

If a line AB intersects y-axis at $(0, b)$, then b is called y-intercept of the line AB.



Derivations of Standard Forms of Equations of Straight Lines

1. Slope Intercept Form

Theorem: Equation of straight line (non-vertical) with slope (gradient) m and y-intercept c is:

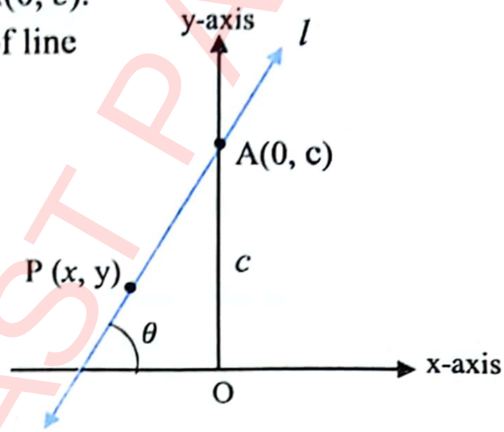
$$y = mx + c$$

Proof:

Let P(x, y) be any point on the straight line l with slope m and y-intercept c.

y-intercept c means that line intersects y-axis at point A(0, c).

As P(x, y) and A(0, c) lie on the line l, therefore slope of line is:



$$m = \frac{y - c}{x - 0}$$

$\Rightarrow \quad y - c = mx$

$\Rightarrow \quad y = mx + c$

Which is equation of line l in slope intercept form.

If c = 0, then the line $y = mx$ passes through origin.

2. Point Slope Form

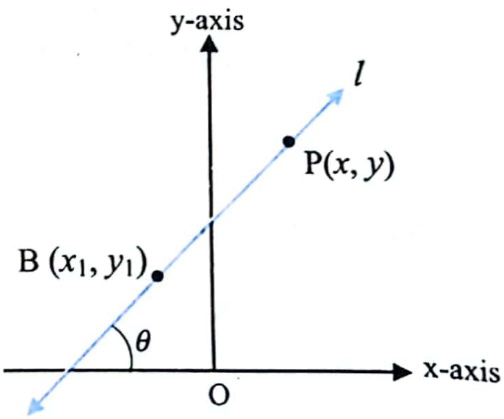
Theorem: Equation of non-vertical straight line passing through point B(x_1, y_1) with slope m is:

$$y - y_1 = m(x - x_1)$$

Proof:

Let P(x, y) be any point on the straight line l with slope m and passing through point B(x_1, y_1).

As P(x, y) and B(x_1, y_1) lie on the line l, therefor slope m of the line is:



$$m = \frac{y - y_1}{x - x_1}$$

$\Rightarrow \quad y - y_1 = m(x - x_1)$

Which is equation of line l in point slope form.

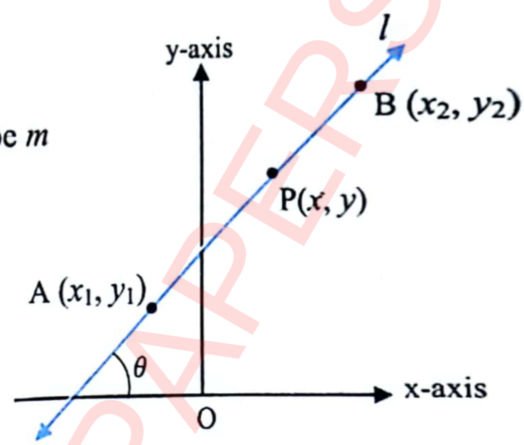
3. Two-Point Form

Theorem: Equation of non-vertical straight line passing through two points A(x_1, y_1) and B(x_2, y_2) is:

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \quad \text{or} \quad y - y_2 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_2)$$

Proof:

Let P(x, y) be any point on the straight line l passing through points A(x_1, y_1) and B(x_2, y_2).
As A(x_1, y_1) and B(x_2, y_2) lie on the line l, therefor slope m of the line is:



$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Now using point slope form of line, we have:

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

or $$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

Which is equation of line l in two-point form.

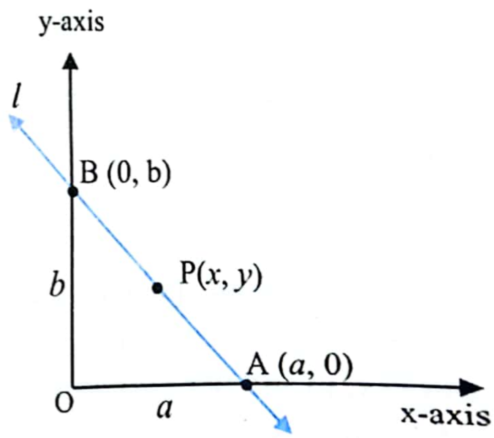
4. Two-Intercept Form

Theorem: Equation of straight line having x-intercept a and y-intercept b is:

$$\frac{x}{a} + \frac{y}{b} = 1$$

Proof:

Let P(x, y) be any point on the straight line l with x-intercept a and y-intercept b.
Obviously, A$(a, 0)$ and B$(0, b)$ lie on the line l.
Therefor slope m of line is:



$$m = \frac{b - 0}{0 - a}$$

Now using point slope form of line, we have:

$$y - b = \frac{b - 0}{0 - a}(x - 0)$$

$$\Rightarrow \quad -ay + ab = bx$$

$$\Rightarrow \quad bx + ay = ab$$

$$\Rightarrow \quad \frac{x}{a} + \frac{y}{b} = 1 \qquad \text{(dividing both sides by } ab\text{)}$$

Which is equation of line l in 2 intercept form.

5. Symmetric Form

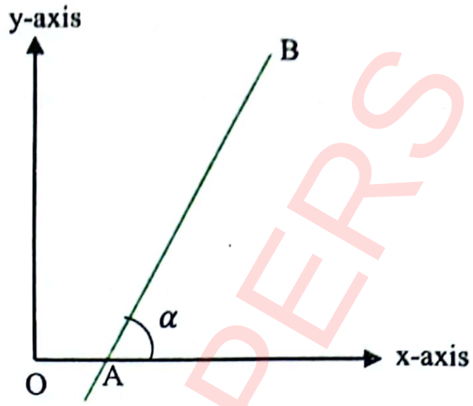
If α is inclination of a line l, then slope of line is:

$$m = \tan\alpha$$

$$\Rightarrow \quad \tan\alpha = \frac{y - y_1}{x - x_1}$$

$$\Rightarrow \quad \frac{\sin\alpha}{\cos\alpha} = \frac{y - y_1}{x - x_1}$$

$$\Rightarrow \quad \frac{x - x_1}{\cos\alpha} = \frac{y - y_1}{\sin\alpha}$$



Which is symmetric form of equation.

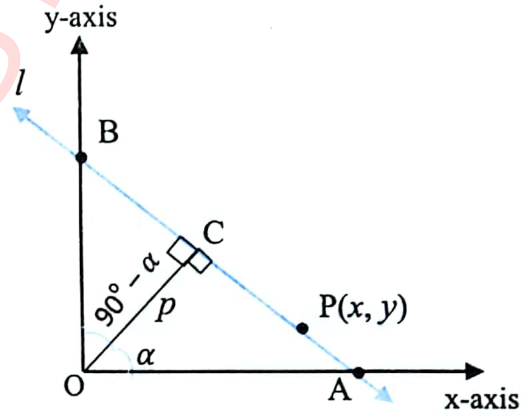
6. Normal Form

Theorem: If p is perpendicular from line l to the origin and α is the inclination of this perpendicular then:

$$x\cos\alpha + y\sin\alpha = p$$

Proof:

Let the line l intersects x-axis and y-axis at points A and B respectively. Let $P(x, y)$ be any point on the line l and OC be perpendicular to line from origin, then $OC = p$



From right angled triangle OCA:

$$\cos\alpha = \frac{OC}{OA} = \frac{p}{OA}$$

or $\quad OA = \dfrac{p}{\cos\alpha}$

From right angled triangle OCB:

$$\text{Cos}\,(90^\circ - \alpha) = \frac{OC}{OB} = \frac{p}{OB}$$

$\sin\alpha = \dfrac{p}{OB}$ or $OB = \dfrac{p}{\sin\alpha}$

As, OA and OB are x-intercept and y-intercept of AB, therefore using two-intercept form, we have:

$$\frac{x}{OA} + \frac{y}{OB} = 1 \Rightarrow \frac{x}{p/\cos\alpha} + \frac{y}{p/\sin\alpha} = 1 \Rightarrow x\cos\alpha + y\sin\alpha = p$$

Which is normal form of equation of line l.

Example 4:

Find the equation of straight line with the following information:

(i) slope = 5, y-intercept = 3

(ii) through (5, 0) with slope −3

(iii) through (2, 4) and (−1, −4)

(iv) x- intercept = 6, y-intercept = −4

Solution:

(i) Slope $= m = 5$, y-intercept $= c = 3$

Equation of line in slope intercept form is: $y = mx + c$

$\Rightarrow \quad y = 5x + 3$ or $5x - y + 3 = 0$

(ii) $(x_1, y_1) = (5, 0)$, $m = -3$

Equation of line in point slope form is: $y - y_1 = m(x - x_1)$

Substituting for (x_1, y_1) and m, we have:

$y - 0 = -3(x - 5) \quad \Rightarrow \quad y = -3x + 15 \quad$ or $\quad 3x + y - 15 = 0$

(iii) $(x_1, y_1) = (2, 4)$, $(x_2, y_2) = (-1, -4)$

Equation of line in two-point form is:

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

Substituting the values, we have:

$$\frac{y - 4}{-4 - 4} = \frac{x - 2}{-1 - 2} \quad \Rightarrow \quad \frac{y - 4}{-8} = \frac{x - 2}{-3}$$

$$-3(y - 4) = -8(x - 2) \quad \Rightarrow \quad -3y + 12 = -8x + 16$$

$$\Rightarrow 8x - 3y - 4 = 0$$

(iv) x- intercept $= a = 6$, y-intercept $= b = -4$

Equation of line in two-intercepts form is:

$$\frac{x}{a} + \frac{y}{b} = 1$$

Substituting the values, we have:

$$\frac{x}{6} + \frac{y}{-4} = 1 \quad \Rightarrow \quad -4x + 6y = -24$$

or $\quad 2x - 3y - 12 = 0$

Check Point

1. Find equation of line with slope -2, passing through **(3, 0)**.
2. Find equation of line passing through (0, 2) and (4, 0).

Example 5:

Find the equation of straight line:

(i) through $(3, -2)$ and perpendicular to the line with slope 3.

(ii) having y-intercept $= -7$ and parallel to the line with slope 4.

(iii) through $(-5, 1)$ and perpendicular to the line passing through $(0, 9)$ and $(-2, 6)$.

(iv) if length of perpendicular from the origin is 6 units inclined at 60°.

Solution:

(i) $(x_1, y_1) = (3, -2)$

Slope of given line = 3

As, the required line is perpendicular to the given line, therefore, slope of required line is:

$m = -\frac{1}{3}$

$\therefore$ Equation of required line is:

$$y + 2 = -\frac{1}{3}(x - 3) \qquad \text{(point-slope form)}$$

$$\Rightarrow \quad 3y + 6 = -x + 3 \quad \Rightarrow \quad x + 3y + 9 = 0$$

(ii) y-intercept = −7

Slope of given line = 4

As, the required line is parallel to the given line, therefore, slope of required line is:

$m = 4$

$\therefore$ Equation of required line is:

$$y = 4x - 7 \qquad \text{(slope intercept form)}$$

$$\Rightarrow \quad 4x - y - 7 = 0$$

(iii) $(x_1, y_1) = (-5, 1)$ and perpendicular the line passing through (0, 9) and (−2, 6).

$$\text{Slope of given line} = \frac{6-9}{-2-0} = \frac{-3}{-2} = \frac{3}{2}$$

As, the required line is perpendicular to the given line, therefore, slope of required line is:

$m = -\frac{2}{3}$

$\therefore$ Equation of required line is:

$$y - 1 = -\frac{2}{3}(x + 5) \qquad \text{(point-slope form)}$$

$$\Rightarrow \quad 3y - 3 = -2x - 10 \quad \Rightarrow \quad 2x + 3y + 7 = 0$$

(iv) $p = 6$ and $\theta = 60^\circ$

Equation of line in normal form is:

$$x\cos 60^\circ + y\sin 60^\circ = 6$$

$$\Rightarrow \quad x \times \frac{1}{2} + y \times \frac{\sqrt{3}}{2} = 6$$

$$\Rightarrow \quad x + \sqrt{3}\,y = 12 \quad \text{or} \quad x + \sqrt{3}\,y - 12 = 0$$

General Form of Equation of Straight Line

General form of a linear equation in two variables x and y is:

$$ax + by + c = 0$$

where a, b and c are real numbers and, a and b cannot be both zero.

Theorem:

A linear equation $ax + by + c = 0$, in two variables x and y represents a straight line. Here a, b, c are constants, and a and b are not simultaneously zero.

Proof:

$$ax + by + c = 0 \qquad (1)$$

Case 1: When $a = 0$ and $b \neq 0$ then equation (1) takes the form.

$$by + c = 0 \quad \text{or} \quad y = -\frac{c}{b}$$

which represents equation of line parallel to x-axis.

Case 2: When $a \neq 0$ and $b = 0$ then equation (1) takes the form.

$$ax + c = 0 \quad \text{or} \quad x = -\frac{c}{a}$$

which represents equation of line parallel to y-axis.

Case 3: When $a \neq 0$ and $b \neq 0$ then equation (1) takes the form.

$$by = -ax - c = 0 \quad \text{or} \quad y = -\frac{a}{b}x - \frac{c}{b} = (-\frac{a}{b}x) + (-\frac{c}{b})$$

which represents slope-intercept form of equation of line with:

slope $= -\frac{a}{b}$ and y-intercept $= -\frac{c}{b}$

Key Fact

From case 3, we can have a rule for finding slope of equation of line $ax + by + c = 0$ as follows:

$$\text{slope} = -\frac{a}{b} = -\frac{\text{coefficient of } x}{\text{coefficient of } y}$$

Reduction of General Form of Equation of a Straight Line to Other Standard Forms

We know that general form of equation of straight line in two variable is:

$$ax + by + c = 0 \qquad (1)$$

We can reduce equation (1) into various standard forms as follows:

1. Slope-Intercept Form

From equation (1) we can write.

$$by = -ax - c = 0 \quad \text{or} \quad y = -\frac{a}{b}x - \frac{c}{b} = (-\frac{a}{b}x) + (-\frac{c}{b})$$

which represents slope-intercept form of equation of line with:

slope $= -\frac{a}{b}$ and y-intercept $= -\frac{c}{b}$

2. Point-Slope Form

Slope of equation (1) is $m = -\frac{a}{b}$

Check Point

Find equation of sides of triangle with vertices P(1, 2), Q(5, 6) and R(11, 12).

A point on equation (1) is $(x_1, y_1) = (0, -\frac{c}{b})$.

∴ Equation of line in point-slope form becomes:

$$y - \frac{c}{b} = -\frac{a}{b}(x - (-\frac{c}{b})) \quad \text{or} \quad y - \frac{c}{b} = -\frac{a}{b}(x + \frac{c}{b})$$

3. Two-Point Form

Two points on equation (1) can be taken as $(x_1, y_1) = (0, -\frac{c}{b})$ and $(x_2, y_2) = (-\frac{c}{a}, 0)$.

∴ Equation of line in two-point form is:

$$\frac{y - \frac{c}{b}}{0 - \frac{c}{b}} = \frac{x - 0}{\frac{c}{a} - 0} \quad \text{or} \quad y - \frac{c}{b} = -\frac{a}{b}(x - 0)$$

4. Two-Intercept Form

From equation (1)

$$ax + by = -c \quad \text{or} \quad \frac{ax}{-c} + \frac{by}{-c} = 1 \quad \text{or} \quad \frac{x}{-c/a} + \frac{y}{-c/b} = 1$$

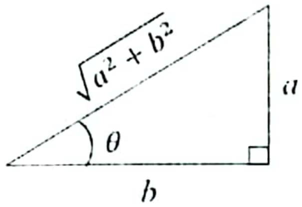
which is equation in two-intercept form with:

x-intercept $= -\frac{c}{a}$ and y-intercept $= -\frac{c}{b}$

5. Symmetric Form

Slope of equation (1) is $m = \tan\theta = -\frac{a}{b}$

$\text{Sin}\theta = \frac{a}{\sqrt{a^2+b^2}}$ and $\text{Cos}\theta = \frac{b}{\sqrt{a^2+b^2}}$



A point on equation (1) is $(x_1, y_1) = (0, -\frac{c}{b})$.

∴ Equation of line in symmetric form is:

$$\frac{x - x_1}{\text{Cos}\theta} = \frac{y - y_1}{\text{Sin}\theta} \quad \text{or} \quad \frac{x - 0}{\frac{b}{\sqrt{a^2+b^2}}} = \frac{y + \frac{c}{b}}{\frac{a}{\sqrt{a^2+b^2}}}$$

6. Normal Form

Equation in normal form is:

$$x\cos\theta + y\sin\theta = p \qquad (2)$$

From equation (1), we have:

$$ax + by = -c \qquad (3)$$

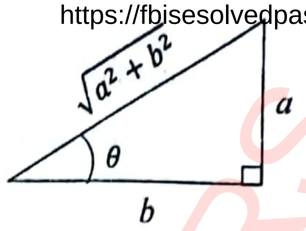
Equations (2) and (3) are identical, therefore:

$$\frac{a}{\cos\alpha} = \frac{b}{\sin\alpha} = \frac{-c}{p}$$

$$\Rightarrow \quad \frac{p}{-c} = \frac{\cos\theta}{a} = \frac{\sin\theta}{b} = \frac{\sqrt{\cos^2\theta + \sin^2\theta}}{\pm\sqrt{a^2+b^2}} = \frac{1}{\pm\sqrt{a^2+b^2}}$$

$$\Rightarrow \quad p = \frac{-c}{\pm\sqrt{a^2+b^2}}$$

Now, dividing both sides by $\pm\sqrt{a^2+b^2}$, we get



$$\frac{ax}{\pm\sqrt{a^2+b^2}} + \frac{by}{\pm\sqrt{a^2+b^2}} = \frac{-c}{\pm\sqrt{a^2+b^2}}$$

$$\Rightarrow \quad x\left(\frac{a}{\pm\sqrt{a^2+b^2}}\right) + y\left(\frac{b}{\pm\sqrt{a^2+b^2}}\right) = \frac{-c}{\pm\sqrt{a^2+b^2}}$$

Which is required normal form of equation of line.

The sign of radical is choosen in this way that right hand side becomes positive.

Example 6:

Reduce the equation $6x - 5y + 15 = 0$ into:

(i) slope-intercept form (ii) two-intercept form (iii) point-slope form
(iv) two-point form (v) normal form (vi) symmetric form

Solution:

(i) $6x - 5y + 15 = 0 \Rightarrow 5y = 6x + 15 \Rightarrow y = \frac{6}{5}x + 3$

where $m = \frac{6}{5}$ and $c = 3$

(ii) $6x - 5y + 15 = 0 \Rightarrow 6x - 5y = -15$

$$\Rightarrow \left(\frac{6x}{-15}\right) - \left(\frac{5y}{-15}\right) = 1 \Rightarrow \frac{x}{-15/6} + \frac{y}{15/5} = 1$$

$$\Rightarrow \frac{x}{-5/2} + \frac{y}{3} = 1$$ where x-intercept $= -\frac{5}{2}$ and y-intercept $= 3$

(iii) A point on the line is $(x_1, y_1) = (0, 3)$. Also slope of line is $= m = -\left(\frac{6}{-5}\right) = \frac{6}{5}$

Equation of line can be written as $y - 3 = \frac{6}{5}(x - 0)$

(iv) Two points on the line are $(x_1, y_1) = (0, 3)$ and $(x_2, y_2) = (\frac{-5}{2}, 0)$

Equation of line passing through both points is:

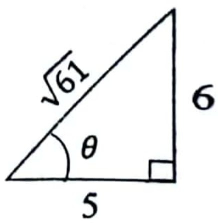
$$\frac{y-3}{0-3} = \frac{x-0}{\frac{-5}{2}-0}$$

(v) Given equation can be written as $6x - 5y = -15$

Or $-6x + 5y = 15$ (Make right side positive)

Dividing both sides of $-6x + 5y = 15$ by $\sqrt{(-6)^2 + 5^2} = \sqrt{36 + 25} = \sqrt{61}$

$$\frac{-6}{\sqrt{61}}x + \frac{5}{\sqrt{61}}y = \frac{15}{\sqrt{61}} \quad \text{or} \quad x\left(\frac{-6}{\sqrt{61}}\right) + y\left(\frac{5}{\sqrt{61}}\right) = \frac{15}{\sqrt{61}}$$



Which is normal from of equation.

(vi) Given equation is $6x - 5y + 15 = 0$.

$m = \tan\theta = \frac{6}{5}$ which implies $\sin\theta = \frac{6}{\sqrt{61}}$ and $\cos\theta = \frac{5}{\sqrt{61}}$

A point on the line is $(x_1, y_1) = (0, 3)$.

Thus equation in symmetric form is:

$$\frac{x-0}{\frac{5}{\sqrt{61}}} = \frac{y-3}{\frac{6}{\sqrt{61}}}$$

EXERCISE 8.2

1. Find the equation of horizontal line passing through:

   (i) (2, 3) (ii) (8, 0) (iii) (−5, −9) (iv) $\left(\frac{-3}{2}, \frac{-5}{2}\right)$

2. Find the equation of vertical line passing through:

   (i) (1, 5) (ii) (9, 6) (iii) (−4, −7) (iv) $\left(\frac{1}{4}, \frac{3}{4}\right)$

3. Find equation of the line with the following information:
   (i) slope = 2, y-intercept = −3
   (ii) through (−5, 7) with slope 4
   (iii) through (4, −5) with slope 0
   (iv) through (−2, 9) with slope undefined
   (v) through (−6, 1) and (2, −4)
   (vi) through (2, −4) and (8, 4)
   (vii) x-intercept = −6, y-intercept = 5
   (viii) slope = −1, x-intercept = 11

4. Find equation of line in symmetric form when:

   (i) $(x_1, y_1) = (-4, 2)$ and $\tan\theta = \frac{3}{4}$ (ii) $(x_1, y_1) = (6, -6)$ and $\theta = 30^\circ$

5. Find equation of line in normal form when:

   (i) $p = 5$ and $\theta = 120^\circ$ (ii) $p = 10$ and $\tan\theta = 1$

6. Find the equation of straight line:
   (i) through (−4, −4) and parallel to the line with slope −5.
   (ii) through (5, −1) and perpendicular to the line with slope $\frac{1}{4}$.
   (iii) having y-intercept = 4 and parallel to the line with slope $\frac{1}{2}$.
   (iv) having x-intercept = −2 and perpendicular to the line with slope 4.
   (v) through (−1, 4) and perpendicular to the line passing through (3, 0) and (1, −2).
   (vi) through (6, −4) and parallel to the line passing through (−5, 2) and (3, 6).
7. Find equation of the line through (3, 7) and parallel to the line $4x - 3y + 1 = 0$.
8. Find equation of the line through (−2, −1) and perpendicular to the line $x - 2y = 0$.
9. Find equation of perpendicular bisector of line segment joining (0, 6) and (2, −2).
10. Find equation of medians and altitudes of triangle with vertices

    A(0, 4), B(4, 6) and C(−2, −2).

11. Reduce the equations (a) $6x + 8y - 11 = 0$ (b) $4x - 3y + 9 = 0$ into:

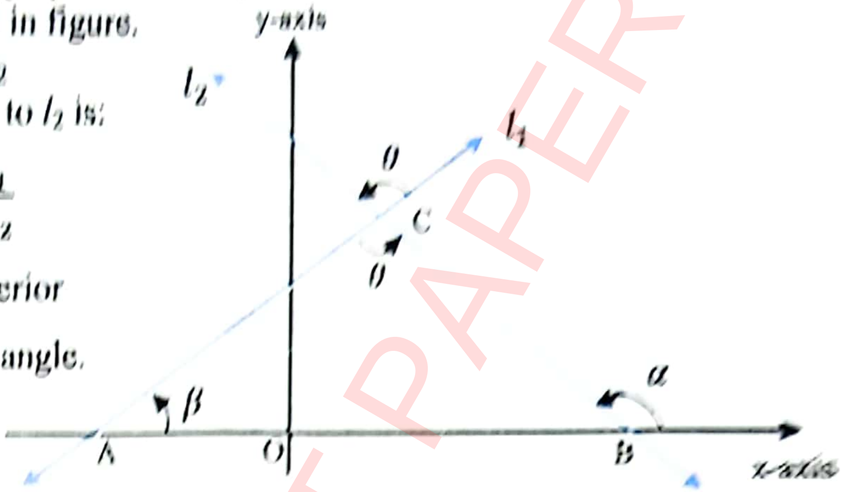
    (i) slope-intercept form (ii) two-intercept form (iii) point-slope form
    (iv) two-point form (v) normal form (vi) symmetric form

Theorem

Let l_1 and l_2 be two non-vertical non-perpendicular coplanar lines making angles α and β with positive x-axis respectively as shown in figure.

If m_1 and m_2 are the slopes of l_1 and l_2 respectively, then the angle θ from l_1 to l_2 is:

$$\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}$$



Proof: In ΔABC, β and θ are two interior angles and α is non adjacent exterior angle.

$\therefore \quad \alpha = \beta + \theta$

$\Rightarrow \quad \theta = \alpha - \beta$

$\Rightarrow \quad \tan\theta = \tan(\alpha - \beta)$

$\Rightarrow \quad \tan\theta = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

As $\tan\alpha = m_2$ and $\tan\beta = m_1$,

$\therefore \quad \tan\theta = \dfrac{m_2 - m_1}{1 + m_2 m_1}$

Key Fact

- If $l_1 \parallel l_2$, then $\tan\theta = 0$
  $\Rightarrow \dfrac{m_2 - m_1}{1 + m_2 m_1} = 0 \Rightarrow m_1 = m_2$
- If $l_1 \perp l_2$, then $\tan\theta = \tan 90^\circ = \infty$
  $\Rightarrow \dfrac{m_2 - m_1}{1 + m_2 m_1} = \infty$
  $\Rightarrow 1 + m_2 m_1 = 0 \Rightarrow m_2 m_1 = -1$
- Angles are measured from positive x-axis in counter clock-wise direction.

Example 7:

Find the measure of angle from l_1 to l_2 if: slope of $l_1 = -1$ and slope of $l_2 = 2$

Solution:

slope of $l_1 = m_1 = -1$ and slope of $l_2 = m_2 = 2$

If θ be the angle from l_1 to l_2 then:

$$\tan\theta = \frac{m_2 - m_1}{1 + m_2 m_1} = \frac{2 - (-1)}{1 + 2(-1)} = -3$$

$$\theta = \tan^{-1}(-3) = 108.43^\circ$$

Example 8:

Find the interior angles of triangle whose vertices are A(–2, 0), B(3, 0) and C(6, 5).

Solution:

Let the slopes of sides AB, BC and AC be denoted by m_1, m_2 and m_3 respectively.

$\therefore \; m_1 = \dfrac{0-0}{3+2} = 0, \quad m_2 = \dfrac{5-0}{6-3} = \dfrac{5}{3}, \quad m_3 = \dfrac{5-0}{6+2} = \dfrac{5}{8}$

Let α, β and γ be interior angles of ΔABC at vertex A, B and C respectively, then:

Angle is measured from AB to AC.

$$\tan\alpha = \frac{m_3 - m_1}{1+m_3m_1} = \frac{\frac{5}{8} - 0}{1+\left(\frac{5}{8}\right)(0)} = \frac{5}{8}$$

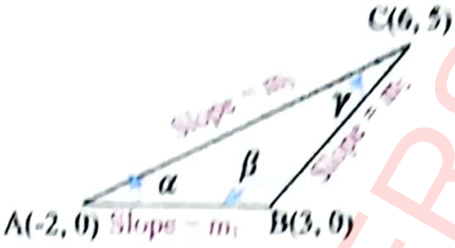
$$\Rightarrow \alpha = \tan^{-1}\left(\frac{5}{8}\right) = 32^\circ$$

$$\tan\beta = \frac{m_1 - m_2}{1+m_1m_2} = \frac{0-\frac{5}{3}}{1+(0)\left(\frac{5}{3}\right)} = -\frac{5}{3}$$

$$\Rightarrow \beta = \tan^{-1}\left(-\frac{9}{19}\right) = 121^\circ$$

$$\tan\gamma = \frac{m_2 - m_3}{1+m_2m_3} = \frac{\frac{5}{3}-\frac{5}{8}}{1+\left(\frac{5}{3}\right)\left(\frac{5}{8}\right)} = \frac{\frac{25}{24}}{\frac{49}{24}} = \frac{25}{49}$$

$$\Rightarrow \gamma = \tan^{-1}\left(-\frac{9}{7}\right) = 27^\circ$$



Key Fact

It is better to find third angle of triangle by subtracting sum of measures of first two angles from 180°.

Point of Intersection of Two Straight Lines

Let l_1 and l_2 be two non-parallel straight lines such that:

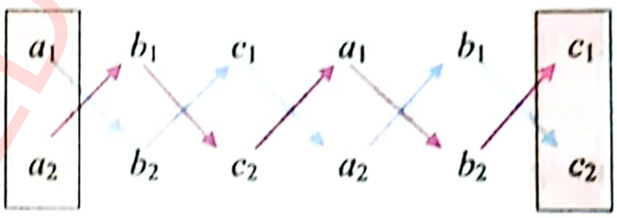
$$l_1 : a_1x + b_1y + c_1 = 0 \qquad \text{and} \qquad l_2 : a_2x + b_2y + c_2 = 0$$

where $a_1b_2 - a_2b_1 \neq 0$ otherwise $l_1 \parallel l_2$.

Let $P(x_1, y_1)$ be the point of intersection of l_1 and l_2, then:

$$a_1x_1 + b_1y_1 + c_1 = 0 \qquad (1)$$

$$a_2x_1 + b_2y_1 + c_2 = 0 \qquad (2)$$



Solving (1) and (2) simultaneously, we get:

$$\frac{x_1}{b_1c_2 - b_2c_1} = \frac{y_1}{a_2c_1 - a_1c_2} = \frac{1}{a_1b_2 - a_2b_1}$$

$$\Rightarrow x_1 = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} \qquad \text{and} \qquad y_1 = \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}$$

Therefore, $\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}\right)$ is required point of intersection.

Example 9:

Find point of intersection of following lines.

$$x + 2y = 3 \qquad \text{(i)}$$

$$3x - y = 2 \qquad \text{(ii)}$$

Solution:

Multiplying equation (ii) by 2 we get:

Key Fact

- Solution of two parallel lines does not exist.
- Two non-parallel lines intersect at one and only one point.
- Infinite number of lines can pass through a point.

$6x - 2y = 4$ (iii)

Adding (i) and (iii), we have:

$7x = 7$ or $x = 1$

Substituting $x = 1$, in equation (i), we get:

$1 + 2y = 3 \Rightarrow 2y = 2$ or $y = 1$

$\therefore$ Required point of intersection is (1, 1).

Equation of Family of Lines

Let l_1 and l_2 be two non-parallel straight lines:

$$l_1 : a_1x + b_1y + c_1 = 0 \quad \text{(i)}$$

$$l_2 : a_2x + b_2y + c_2 = 0 \quad \text{(ii)}$$

with point of intersection $P(x_1, y_1) = P\left(\dfrac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \dfrac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}\right)$ where $a_1b_2 - a_2b_1 \neq 0$.

We can find a family of lines through the point of intersection P.

For a non-zero k, the equation:

$$a_1x + b_1y + c_1 + k(a_2x + b_2y + c_2) = 0 \quad \text{(iii)}$$

is also linear and represents a straight line.

Equation (iii) represents infinite number of lines for different values of k and therefore it is known as family of lines.

It can be easily proved that if (x_1, y_1) is point of intersection of (i) and (ii), then equation (iii) also passes through (x_1, y_1).

Thus equation (iii) is the required family of lines passing through point of intersection of equations (i) and (ii). As mentioned above that equation (iii) represents infinite number of lines by changing values of k. For particular value of k, a particular line of family from (iii) can be determined by imposing one more condition.

Example 10:

Find equation of line passing through point of intersection of lines $x + 2y = 0, x - y = 3$

(a) passing through point (1, 1) (b) parallel to $3x - 4y + 7 = 0$

Solution:

$$x + 2y = 0 \quad \text{(i)}$$

$$x - y - 3 = 0 \quad \text{(ii)}$$

A family of lines through the point of intersection of lines (i) and (ii) is:

$$x + 2y + k(x - y - 3) = 0$$

$\Rightarrow \quad (1 + k)x + (2 - k)y - 3k = 0 \qquad$ (iii)

(a) As equation (iii) passes through (1, 1), therefore:

$$(1 + k)(1) + (2 - k)(1) - 3k = 0$$

$\Rightarrow \quad 1 + k + 2 - k - 3k = 0 \quad \Rightarrow \quad 3 - 3k = 0 \quad \Rightarrow \quad k = 1$

Substituting the value of k in equation (iii), we get the required line as follows:

$$(1 + 1)x + (2 - 1)y - 3 \times 1 = 0 \quad \Rightarrow \quad 2x + y - 3 = 0$$

(b) Slope of line (iii) is:

$$m_1 = -\left(\frac{1+k}{2-k}\right)$$

$3x - 4y + 7 = 0 \qquad$ (iv)

Slope of line (iv) is:

$$m_2 = -\left(\frac{3}{-4}\right) = \frac{3}{4}$$

It is given that lines (iii) and (iv) are parallel, therefore:

$$-\left(\frac{1+k}{2-k}\right) = \frac{3}{4} \quad \Rightarrow \quad \frac{-1-k}{-2+k} = \frac{3}{4} \quad \Rightarrow \quad \frac{-1-k}{-2+k} = \frac{3}{4}$$

$$4(-1 - k) = 3(-2 + k) \quad \Rightarrow \quad -4 - 4k = -6 + 3k$$

$$7k = 2 \quad \text{or} \quad k = \frac{2}{7}$$

Substituting the value of k in $x + 2y + k(x - y - 3) = 0$.

$$x + 2y + \frac{2}{7}(x - y - 3) = 0 \quad \Rightarrow \quad 7x + 14y + 2x - 2y - 6 = 0$$

$\Rightarrow \quad 9x + 12y - 6 = 0$

Which is required line.

EXERCISE 8.3

1. Find the measure of angle from l_1 to l_2 if:
   (i) slope of $l_1 = 0$ and slope of $l_2 = 1$ (ii) slope of $l_1 = -0.5$ and slope of $l_2 = 4.5$
   (iii) slope of $l_1 = \tan 45^\circ$ and slope of $l_2 = \tan 135^\circ$
2. Find the measure of angle from l_1 to l_2 if:
   (i) l_1: joining (2, 0) and (5, 0) $\quad$ l_2: joining (2, 0) and (5, 5)
   (ii) l_1: joining (–2, 1) and (3, 4) $\quad$ l_2: joining (–1, 3) and (4, 8)
   (iii) l_1: joining (–5, –4) and (5, 1) $\quad$ l_2: joining (–3, 2) and (0, 5)
   (iv) l_1: joining (2, –6) and (5, –9) $\quad$ l_2: joining (5, –5) and (–10, –5)

(v) l_1: joining (0, –3) and (7, –9) l_2: joining (2, –2) and (–8, –12)

3. Find the interior angles of triangle ABC when:
   (i) Slope of AB = 0, Slope of BC = –1, Slope of AC = 1
   (ii) Slope of AB = 0.25, Slope of BC = 1.25, Slope of AC = 1
   (iii) Slope of AB = 0.4, Slope of BC = –1.5, Slope of AC = 1.667
   (iv) Slope of AB = –1, Slope of BC = 0.8, Slope of AC = 0

4. Find angle between lines:
   (i) $3x+2y+5=0$ and $2x-3y+8=0$
   (ii) $x+2y-6=0$ and $2x-4y+9=0$
   (iii) $6x-y+1=0$ and $x-7y+12=0$

5. Find the interior angles of triangle XYZ whose vertices are:
   (i) X(–2, 3), Y(–3, – 4), Z(5, 2) (ii) X(–3, 2), Y(0, –1), Z(3, 3)
   (iii) X(–2, 0), Y(1, – 4), Z(6, 6) (iv) X(–4, 1), Y(0, –3), Z(4, 3)

6. Find the point of intersection of lines:
   (i) $2x+y+1=0$ and $x-y-4=0$
   (ii) $x+y+3=0$ and $2x-5y+8=0$
   (iii) $2x+5y+3=0$ and $3x-4y-5=0$

7. Find equation of line passing through point of intersection of lines
   $3x+2y+1=0$, $x-2y+3=0$ and
   (a) passing through point (–1, 0). (b) parallel to $3x-4y+3=0$.

8. Find the equation of family of lines passing through point of intersection of
   $6x+5y+3=0$ and $2x-5y+13=0$ with slope 3.

9. Find equation of line passing through point of intersection of lines
   $2x-5y+4=0$, $6x-4y+5=0$ and
   (a) parallel to x-axis. (b) parallel to y-axis.

10. Find equation of line passing through point of intersection of lines
    $2x-y+2=0$, $x-2y+1=0$ and
    (a) parallel to $x-2y+11=0$ (b) perpendicular to $2x+5y+2=0$.

11. Find equation of line passing through point of intersection of lines
    $x-2y+4=0$, $3x-y-3=0$ and
    (a) parallel to line passing through (2, –3) and (0, 4).
    (b) perpendicular to line passing through (2, –3) and (0, 4).

Real World Problems of Coordinate Geometry

Example 11: In the linear equation $y = 1.12x + 8$ if "x" represents the number of kilometers and "y" represents the cost of the bus fare.

(i) What will be the cost of bus fare after travelling 50 km?

(ii) Asif paid a bus fare of Rs. 480 in a journey. Find number of kilometers travelled by him?

Solution:

Given: $y = 1.12x + 8$ .... (i)

(i) Substituting $x = 50$ in equation (i), we have:

$$y = 1.12 \times 50 + 8 = 64$$

$\therefore$ Cost of bus fare = Rs.64

(ii) Substituting $y = 480$ in equation (i), we have:

$$480 = 1.12\,x + 8 \quad \Rightarrow \quad 1.12\,x = 480 - 8 = 472$$

$$x = 472 \div 1.12 = 421.43$$

$\therefore$ Numbers of kilometers = 421.43

Example 12:

Saadia buys mangoes @ Rs. 150 per kilogram and melon @ Rs. 80 per kilogram. She has Rs. 620 to spend on fruit. Write an equation in standard form that describes the situation. If she buys 2 kilograms of mangoes, how many kilograms of melon can she buy?

Solution:

Suppose x denotes number of kilograms of mangoes and y denotes number of kilograms of melon.

The equation that describes this situation is:

$$150\,x + 80\,y = 620 \qquad \text{(i)}$$

As she buys 2 kilograms of oranges, substituting $x = 2$ into the equation (i) we get:

$$150(2) + 80\,y = 620 \quad \text{or} \quad 300 + 80\,y = 620$$

$$\Rightarrow 80\,y = 620 - 300 = 320 \quad \Rightarrow \quad y = 320 \div 80 = 4$$

$\therefore$ Saadia can buy 4 kilograms of melon.

Example 13:

The pollution index in a large city increases in an approximately linear fashion from 8 am until

3 pm in autumn season. On 24th November, the reading at 10:00 hours is around 80 parts per million (ppm) and at 14:00 hours, the reading is 110 ppm.

(a) Write an equation for this situation.

(b) What does the gradient (slope) mean in terms of pollution?

(c) What does the y-intercept mean in terms of pollution?

(d) What will be the pollution index at 12 pm?

Solution:

(a) If t denotes time and p denotes pollution index then:

$(t_1, p_1) = (10, 80)$ and $(t_2, p_2) = (14, 110)$

Using equation of line in two-point form, we have:

$$\frac{t-t_1}{t_2-t_1} = \frac{p-p_1}{p_2-p_1}$$

Substituting the values, we have:

$$\frac{t-10}{14-10} = \frac{p-80}{110-80} \quad \Rightarrow \quad \frac{t-10}{4} = \frac{p-80}{30}$$

$\Rightarrow \quad 30(t-10) = 4(p-80) \Rightarrow 30t - 300 = 4p - 320$

$\Rightarrow \quad 30t - 4p + 20 = 0 \quad \Rightarrow \quad 15t - 2p + 10 = 0$ ...... (i)

Which is required equation.

(b) From (i), we have:

$2p = 15t + 10 \quad \Rightarrow \quad p = 7.5t + 5$ ...... (ii)

$\Rightarrow \quad m = 7.5$ and $c = 5$

Here the gradient 7.5 means that pollution index is increasing @ 7.5 ppm per hour.

(c) y-intercept = c = 5 means that pollution index at 00:00 hours (12am) is 5 ppm.

You can plot graph to understand in a better way.

(d) Substituting t = 12 in equation (ii), we get:

$p = 7.5 \times 12 + 5 = 95$ ppm

EXERCISE 8.4

1. Nasir sold 'fruiters' @ Rs.120 per dozen and 'Shakri Malta' @ Rs.150 per dozen and earned Rs. 1200. Write an equation in standard form that describes the situation. If he sold 4 dozen of 'Shakri Malta', how many dozens of 'fruiters' he sold?

2. The linear equation $y = 1450x + 2000$ describes the total cost for staying in a hotel for one day, where Rs. 2000 is the rent of room for maximum 8 people and Rs.1450 is the food cost per person.

   (i) Find the cost paid to hotel if a group of 7 people stays for one day.

   (ii) How many people can stay in the hotel for Rs. 13,600 for one day?

3. If one company provides Rs. 5500 per week along with extra bonus of Rs. 700 and the other offers Rs. 800 per day. Convert the data into linear equations and tell which is better deal for two weeks?

4. A man earns Rs. 120 per hour. He has Rs. 500 in addition with him.

   (i) Write a linear equation for the situation and tell how much will he get after 12 hours?

   (ii) What does slope show in this situation?

   (iii) What does y-intercept represent here?

5. Ali shifted in a rented house on first September. The electric meter of house was showing 44 units on that day. If the average electricity consumed is 18 units per day:

   (i) Represent the situation through linear equation.

   (ii) How many units are consumed till 30 September?

   (iii) What will be the bill after one month @ Rs. 20 per unit?

   (iv) After how many days, the meter shows 404 units?

6. Alia hired a taxi with a fixed charge of Rs. 1500 plus Rs. 450 per 30 minutes.

   (i) Represent the relation as a linear equation.

   (ii) What will be the cost of taxi fare after 5 hours?

   (iii) What is the slope of equation in the case?

7. (i) Derive the relation between Fahrenheit and Celsius scales in slope-intercept form.

   (ii) What does y-intercept and slope show in the equation?

   (iii) What is temperature in Fahrenheit when temperature in Celsius is $5^\circ C$?

8. A cricket team scores 96 runs in 16 overs and 180 runs in 30 overs.

   (i) Write an equation of line for this situation.

   (ii) What does the gradient mean in terms of scores?

   (iii) What does the y-intercept mean in terms of scores?

   (iv) What will be the predicted score after 45 overs?

   (v) After how many overs, the predicted score will be 240?

9. Abdullah rented a truck for one day. The truck company charged Rs. 5000 per day and some additional money per kilometre. He drives 125 kilometres and paid Rs. 30,000.
   (i) Write an equation in point-slope form that describes this situation.
   (ii) Find the amount per kilometre the truck rental company charges and relate it with slope.
   (iii) How much would it cost if Abdullah drove 180 km?
10. A ship starts travelling from Karachi with latitude of 25° N and longitude 67° E. If the ship travels in a straight line and reaches a destination with latitude of 32° N and longitude 54° E, then derive the equation of line in point-slope form. If the ship moves to another location with a latitude of 39° N, what is longitude of that location?
11. Length and width of a plot are in the ratio 2 : 1.
   (i) Write equation of line and find the length of plot if the width of plot is 30 feet.
   (ii) What is slope in this case and what does it mean?

KEY POINTS

- The inclination of a straight line is the angle θ which the line makes with positive x- axis measured in anti-clockwise direction.
- Line which is neither horizontal nor vertical is called an oblique line.
- The gradient of a straight line is the tangent of the angle which the line makes with the positive direction of the x-axis.
- Gradient of horizontal line is 0 and that of vertical line is undefined.
- If two lines have same gradients, they are parallel.
- If m_1 and m_2 are gradients of two perpendicular lines, then:

$$m_1 \times m_2 = -1$$

- Three points A, B and C are collinear if gradients of AB, BC and AC are equal.
- In $y = b$, if $b > 0$, then the line is above the x-axis, if $b < 0$, then the line is below the x-axis and $b = 0$, then the line becomes x-axis.
- In $x = a$, if $a > 0$, then the line is on the right of the y-axis, if $a < 0$, then the line is on the left of the y-axis and if $a = 0$, then the line becomes y-axis.
- If a line AB intersects x-axis at $(a, 0)$, then a is called x-intercept of the line AB and if a line AB intersects y-axis at $(0, b)$, then b is called y-intercept of the line AB.

- Equation of straight line with slope m and y-intercept c is $y = mx + c$.
- Equation of straight line passing through point $B(x_1, y_1)$ with slope m is $y - y_1 = m(x - x_1)$.
- Equation of straight line passing through two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is:

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

- Equation of straight-line having x-intercept a and y-intercept b is:

$$\frac{x}{a} + \frac{y}{b} = 1$$

- If p is perpendicular from line l to the origin and α is the inclination of this perpendicular then $x\cos\alpha + y\sin\alpha = p$
- A linear equation $ax + by + c = 0$, in two variables x and y represents a straight line where a, b, c are constants, and a and b are not simultaneously zero.
- Angle between two lines with slopes m_1 and m_2 is defined by:

$$\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}$$

- For a non-zero k, the equation $a_1x + b_1y + c_1 + k(a_2x + b_2y + c_2) = 0$ is also linear and represents family of lines passing through (x_1, y_1).

MISCELLANEOUS EXERCISE 8

1. Encircle the correct option in the following.

(i) Slope of two lies are 0 and ∞. What is angle between both lines?

(a) 0° (b) 60° (c) 90° (d) 180°

(ii) One angle of right triangle ABC is determined by the line joining A(1, 2) and B(2, 3). Find the third angle.

(a) 30° (b) 45° (c) 60° (d) 80°

(iii) Slope of (−1, 6) and (1, y) is 3. What is y?

(a) 10 (b) 11 (c) −12 (d) 12

(iv) The line $5x - ky - 3 = 0$ passes through (1, 2). What is k?

(a) 1 (b) −1 (c) −2 (d) 2

(v) The line $5x - 6 = 0$ represents a line:

(a) parallel to x-axis (b) parallel to y-axis
(b) passing through origin (d) touching both axis

(vi) Line $y = 0$ represents:

(a) x-axis (b) y-axis (c) a plane (d) a point

(vii) The line $y = b$, is above the x-axis, if
(a) $b = 0$ (b) $b < 0$ (c) $b > 0$ (d) $b \neq 0$

(viii) The line $x = a$, is left to the y-axis, if
(a) $a = 0$ (b) $a \neq 0$ (c) $a > 0$ (d) $a < 0$

(ix) Slope of a line l is -4. What is slope of a line perpendicular to l?
(a) $\frac{1}{4}$ (b) $-\frac{1}{4}$ (c) 4 (d) -4

(x) Which of the following line has slope $\frac{2}{3}$?
(a) $2x + 3y = 0$ (b) $2x - 3y = 2$ (c) $3x - 2y = 1$ (d) $3x + 2y = 3$

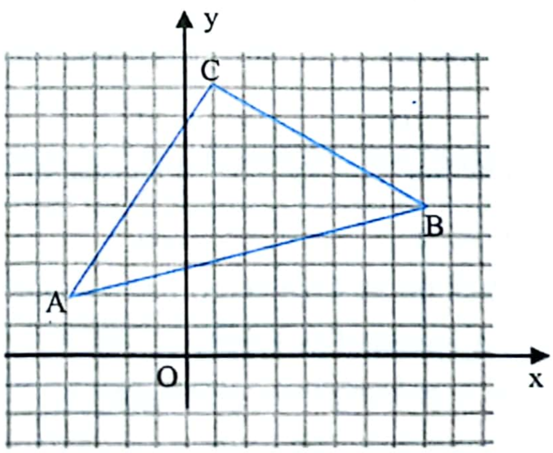
(xi) The line $y = 5x - 3$ is written in the form:
(a) point-slope (b) two-intercept (c) slope-intercept (d) two-point

(xii) x-intercept of the line $x + y = 5$ is:
(a) 1 (b) -1 (c) 5 (d) -5

(xiii) A line intersects both axis at (2, 0) and (0, 7) respectively. Its y-intercept is:
(a) -2 (b) 2 (c) -7 (d) 7

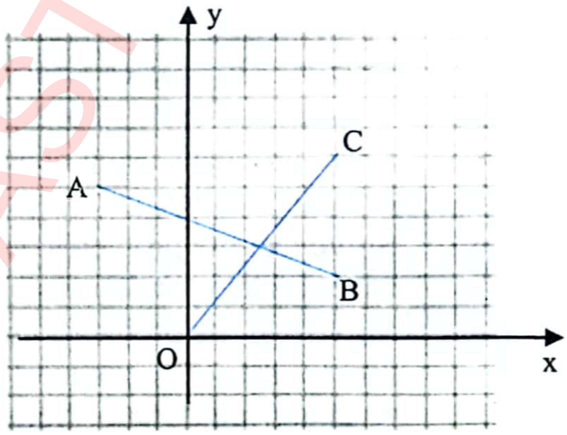
(xiv) Point of intersection of lines $9x - 7y = 0$ and $8x - 11y = 0$, is:
(a) (0 ,0) (b) (0, 7) (c) (9, 0) (d) (7, 9)

2. Prove that A(3, –10), B(1, 4) and C(2, –3) are collinear points.
3. x-intercept of a line is double of y-intercept. Find equation of line if it passes through (2, 1).
4. Reduce $5x - 2y + 1 = 0$ into slope intercept form and two intercept form.
5. (i) Reduce $x - 3y = 3$ into intercept form and find x- and y-intercepts.
(ii) Find its slope and transform the equation into point slope form.
6. Normal form of an equation of line is $x \cos 150^\circ + y \sin 150^\circ = 10$. Transform the equation into slope-intercept form and find its slope and y-intercept.
7. (i) Find vertices of triangle ABC.
(ii) Calculate slopes of its sides.
(iii) Find interior angles of triangle.



8. Two points P(4, −1) and Q(8, 3) lie on a line. Find:
   (i) coordinates of mid-point M of PQ.
   (ii) slope of PM.
   (iii) the equation of line parallel to PQ through (−2, 2).
   (iv) the equation of line perpendicular to PQ through (−2, 2).
9. Two points A(2, −2) and B(4, 6) lie on a line. Find:
   (i) length of AB. (ii) slope of BA.
   (iii) values of a and b when the line AB is $ax + by - 10 = 0$.
   (iv) the equation of line parallel to AB passing through (0, 3).
10. Find equation of line passing through mid-point of (4, 4) and (8, 0) parallel to the line having slope $\frac{2}{5}$.

11. Lines OC and AB are shown in the graph. Find
   (i) coordinates of end points of OC and AB.
   (ii) slopes of both lines.
   (iii) equations of both lines.
   (iv) coordinates of point of intersection of both lines.



12. Locate two points on the coordinate plane that satisfy the equation $x - 2y = 2$. Find:
   (i) the slope of segment l connecting two points.
   (ii) slope of segment p perpendicular to l.
   (iii) mid-point of the segment l.
   (iv) equation of line passing through mid-point of segment l and slope of p.

UNIT
09

GEOMETRY AND POLYGONS

In this unit the students will be able to:

- Differentiate between mathematical statement and its proof.
- Differentiate between an axiom, conjecture and theorem.
- Formulate simple deductive proofs (algebraic proofs that require showing LHS equal to RHS e.g. $(x-3)^2+5=x^2-6x+14$
- Identify similarity of polygons and area and volume of similar figures.
- Solve problems using relationship between areas of similar figures and volumes of similar solids.
- Solve real life problems that involve the properties of regular polygons, triangles and parallelograms.
- Solve real life problems using the following loci and the method of intersecting loci for sets of points in two dimensions which are:
  - at a given distance from a given point.
  - at a given distance from a given straight line.
  - equidistant from two given points.
  - equidistant from two given intersecting straight lines.

Geometry has many uses. It is used whenever we ask questions about the size, shape, volume, or position of an object. As a school subject, it helps to develop logical reasoning. Architects and engineers use geometry in planning buildings, bridges, and roads.

Geometry is used by navigators to guide boats, planes, and even space ships. Military personnel use geometry to guide vessels and aim guns and missiles. Almost everything you do in your daily life involves geometry in some way. We observe many geometrical shapes used in the construction of buildings like masjids, forts and in other buildings.

Demonstrative Geometry

It is a branch of geometry in which geometrical statements are proved through logical reasoning. The reasons can be taken from given information, basic assumptions or already proved results and their corollaries etc.

Reasoning

Reasoning is a way of proving results. The statements without logical reasoning are not acceptable. Different forms of reasoning are accepted in different cases. There are two types of reasoning.

i. Inductive reasoning ii. Deductive reasoning

i. Inductive Reasoning

In inductive reasoning we work from some specific observations to a general result.

For example, if your school starts at 8:00 am daily and you left your home at 7:15 am for school today, you arrived at school on time. So, to arrive at school on time you should leave your home 45 minutes before the school time daily.

Inductive reasoning is commonly used in science. It is not always valid logically because it is not always accurate to assume a general principle to be correct. In above example, perhaps 'today' there was less traffic, and if you leave the house at 7:15 am. On any other day, it might take longer and you might be late for school due to heavy traffic.

ii. Deductive Reasoning

In deductive reasoning we work from some general result to a specific conclusion.

For example, if your school starts at 8:00 am and you leave your home at 7:15 am daily for school, you arrive at school on time. So, to arrive at school on time today you should leave your home 45 minutes before the school time.

Example 1:

Prove that $3x + 9 = 3(x + 3)$

Solution:

Let $x = 1$

LHS $= 3(1) + 9 = 12$, RHS $= 3(1 + 3) = 3(4) = 12$

Similarly, the statement is true for all real numbers.

Hence proved.

Example 2:

Find $(102)^2 = ?$

Solution:

General: $(a + b)^2 = a^2 + b^2 + 2ab$

Check Point

Tell whether the following statement follows inductive or deductive approach?

Statement: If a line segment AB passes through point O, then the line AB (or ray AB) will also pass through point O.

Particular: $(100 + 2)^2 = 100^2 + 2^2 + 2 \times 100 \times 2$
$= 10000 + 4 + 400 = 10404$

Applicability of Deductive Approach

Deductive approach is suitable for giving practice to the student in applying the formula or principles or generalization which has been already arrived at. This method is very useful for fixation and retention of facts and rules as it provides adequate drill and practice.

Mathematical Statement

When we solve any problem in mathematics, our solution is either right or wrong. There is no midway to solve the problems.

A mathematical statement is a meaningful composition of words that can be either true or false. It may contain words and symbols.

Examples:

1. Look at the following sentences:
   (i) Sum of 2 and 5 is 7.
   (ii) Square root of 36 is 6.
   (iii) No line can pass through a point.
   Here the first two sentences are true and third one is false.
   ∴ Above are mathematical statements.
2. $2x + 4 = 2(x + 2)$
   This is a mathematical statement since both sides of the sentence are equal when any real number is substituted for x.
3. Now observe the following sentence.
   (i) Product of numbers x and y is 10.
   We are not sure the statement is true or false as the values of x and y are not known to us.
   ∴ The above statement is an open sentence.
   (ii) If we write the sentence as follows:
   'Product of numbers x and y when $x = 5$ and $y = 2$, is 10'.
   Then the sentence is a mathematical statement (being true).
   (iii) Again, if we write the sentence as follows:
   'Product of numbers x and y when $x = 4$ and $y = 3$, is 10'.
   Then the sentence is again a mathematical statement (being false).

Check Point

Which of the mathematical statements are true?
(a) Square of 12 is 144.
(b) Product of 4, 5 and 8 is 165.
(c) Log 1 = 0
(d) $2^6 = 12$

Proposition

A statement which may or may not be true is called a proposition.

There are three parts of proposition.

(i) The premise

It is an assumption that something is true.

(ii) The argument

The logical chain of reasoning that leads from the premises to the conclusion is known as an argument.

(iii) The conclusion

The result obtained after giving argument to the premises is called conclusion.

The conclusion must be true if the premises are true and the argument is valid.

Key Fact

A mathematical statement consists of two parts. First is the hypothesis or assumptions and the second is the conclusion.

Axiom, Conjecture and Theorem

Fundamental Assumptions

Fundamental assumptions are statements which are regarded true without any proof.

These assumptions play an important role in geometry.

1. Axiom

The word 'Axiom' is derived from the Greek word 'Axioma' which mean 'true without any proof'. Thus, an axiom is a mathematical statement which is assumed to be true without any proof. Axioms are truths that have been derived on the basis of everyday experience and form the basis for all other derivations. Axioms deal with numbers and their relations.

For example:

(i) A number is always equal to itself (reflexive property).
i.e. if x is any real number then $x = x$.
In geometry we can say that if $\overline{AB}$ is any line segment then AB = AB.

(ii) If $a = b$ and $b = c$ then $a = c$ (Transitive property)
Or in geometry, if $\angle A = \angle B$ and $\angle B = \angle C$, then $\angle A = \angle C$.

Key Fact

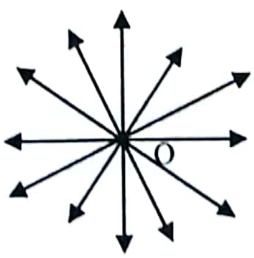
Axioms are also known as properties and first seven axioms are called properties of equations.

2. Postulate

Axioms related to geometry are called postulates.

Some postulates of geometry are given below.

(i) Infinite number of lines can pass through a point.



(ii) Through two different points one and only one line can pass.

(iii) If two points of a line lie in a plane, then whole line lies in that plane.

3. Conjecture

A statement that is believed to be true but its truth has not been proved is called a conjecture. In other words, it is a true statement that needs proof.

For example, observe the following pattern of numbers:

4, 8, 12, 16, ___.

If we are asked, 'what is next number in the pattern?'. We observe that each next number is 4 more than previous one. So, the answer is 20.

∴ One of the conjectures is that, 'the next number is $16 + 4 = 20$'.

Another conjecture could be 'the next number is $16 + 5 - 1 = 20$'.

None of the two has a proof but both conjectures follow simple mathematical rules and axioms.

Key Fact

Conjectures play a very important role in problem-solving in all branches of mathematics including Geometry, where the solution is not always apparent and is generated by following a series of steps. Each of these steps is a 'conjecture' over the previous step'.

Proof

A proof is a series of conjectures and axioms (postulates) and proved theorems that combine together to give a true result.

No assumptions can be made in a mathematical proof. Every step must be proved in the logical sequence. Mathematical proofs use deductive reasoning where a conclusion is drawn from multiple premises. The premises in the proof are called statements.

Theorem

The word theorem is derived from Greek word 'theorien' which means "behold, contemplate or consider."

A mathematical statement which can be proved or supposed provable through logical reasons is called a theorem.

Important Steps of Proof of a Theorem

i. Statement

A description of a geometrical theorem in words should be written first. It is called statement of the theorem.

ii. Figure

After writing the statement a neat figure should be drawn to explain or understand the given information and the result to be proved.

iii. Given

In this step only given information should be written symbolically so that it becomes easy to use in the proof.

iv. Required to Prove

After given information it is necessary to write the result symbolically which is to be proved so that our attention does not divert from the main objective at any stage.

v. Construction

The necessary addition in the figure can be done to prove the theorem easily. This addition in the figure is called construction.

vi. Proof

It is the most important step. It consists of statements and facts along with their reasons through which we obtain the required results.

Corollary

Some results which can be deduced directly from theorems are called corollaries.

Converse of a Theorem

If given and to prove of a theorem are interchanged, the new statement is called converse of a theorem. It is not necessary that converse of a theorem is also a theorem.

EXERCISE 9.1

1. What is the difference between axiom and conjecture?
2. Which of the following are mathematical statements?
   - (i) Difference of 19 and 12 is 7.
   - (ii) $-2+7-3=2$
   - (iii) $34+16 \neq 50$
   - (iv) $a+b=9$
   - (v) $(a+b)^2 = a^2+2ab+b^2$
   - (vi) $2+2\times 2=6$
   - (vii) The product of x and y is smaller than 5.
   - (viii) If x is real then either $x<0$ or $x>0$ or $x=0$.
   - (ix) If $a>b$ and $b>c$ then $a<c$
   - (x) $xy+z=12$
   - (xi) $s-t=4$ if $s=4$ and $t=0$
3. The sum of a and b is equal to 0.
   Is this sentence a mathematical statement? If no! How can we make it a mathematical statement?
4. Prove $(x+1)^2+5=x^2+2x+6$ by taking $x=2$, 5 and 10.
5. Find the next number in the pattern using conjecture.
   1, 3, 7, 15, 31, ____
   State the conjecture used.
6. Which of the following are axioms? How many of the axioms are postulates?
   - (i) If $a=b$ then $b=a$
   - (ii) 2 plus 2 make 4.
   - (iii) One and only one line can pass through two points.
   - (iv) If two sides of a triangle are equal then opposite angles are also equal.

(v) Product of two negative real numbers is always greater than zero.
(vi) All right angles are equal to one another.
(vii) The whole is greater than its part.
(viii) If $a > b$ and $c > d$ then $a + c > b + d$
(ix) It is possible to extend a line segment continuously in both directions.
(x) When we add three consecutive even numbers, their sum is even.

7. Explain all the steps of geometrical proof.

Similarity of Polygons

Similar Figures

Two or more figures that have the same shape but not the same size are called similar **figures.**
Figures (i) and (ii) below represent the pairs of similar figures.

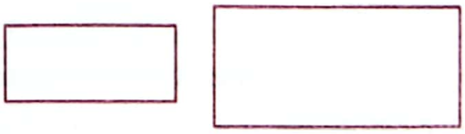


Fig. (i)

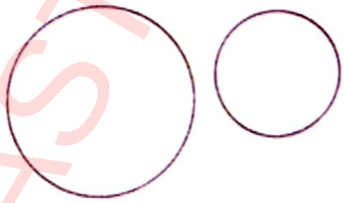


Fig. (ii)

The symbol for similarity is '~'. Thus, if two figures A and B are similar, then we write **A ~ B.**

Example 3: Are the following pairs of figures similar? Explain.

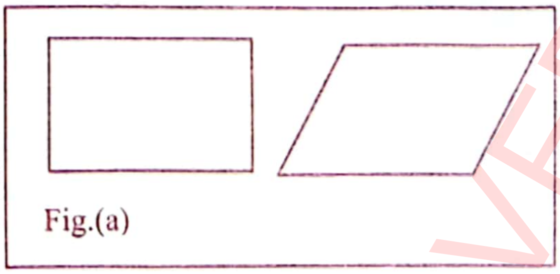


Fig.(a)

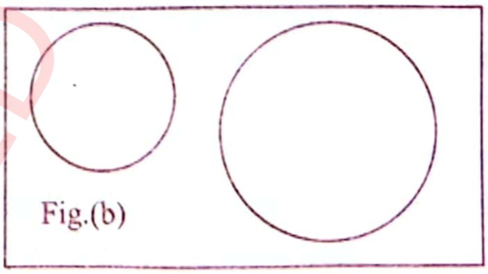


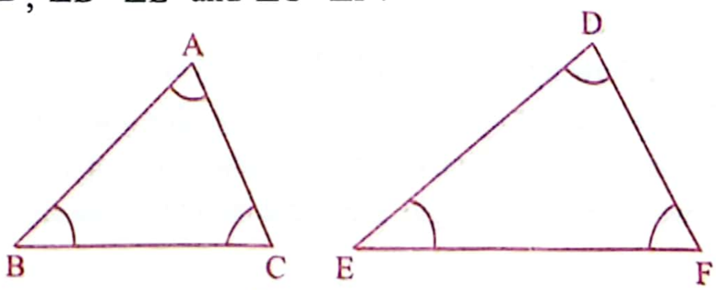
Fig.(b)

Solution:

In Fig.(a), two quadrilaterals do not have the same shape. Hence, they are not similar.
In Fig.(b), two circles have the same shape but not the same size, therefore they are **similar.**

Class Activity 1:

Look at the following triangles ABC and DEF. They have the same shape but not the **same size.**
Therefore ΔABC and ΔDEF are similar.
Now measure corresponding angles of both the triangles.
You will see that: $\angle A = \angle D$, $\angle B = \angle E$ and $\angle C = \angle F$.



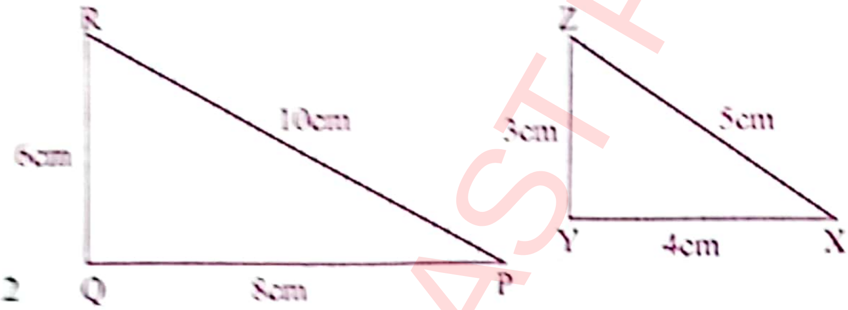
> If two or more figures are similar then their corresponding angles are congruent and vice versa.

Key Fact

(i) If triangles ABC and DEF are similar then, we write $\Delta ABC \sim \Delta DEF$ and read it as "ΔABC is similar to ΔDEF".

(ii) In the above figure if $\angle A = \angle D$, $\angle B = \angle E$ then $\angle C = \angle F$ as the sum of measures of angles of triangle is 180°.

Class Activity 2: Draw the following figures.



In ΔPQR and ΔXYZ

$$\frac{PQ}{XY} = 2, \frac{QR}{YZ} = 2, \frac{PR}{XZ} = 2$$

$$\therefore \frac{PQ}{XY} = \frac{QR}{YZ} = \frac{PR}{XZ}$$

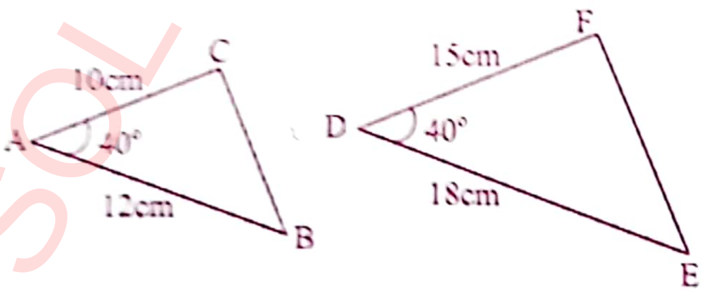
i.e. corresponding sides are proportional.

Hence $\Delta PQR \sim \Delta XYZ$

> If two or more figures are similar then ratios of corresponding sides are equal and vice versa.

Class Activity 3:

Look at the following figures.



In ΔABC and ΔDEF

$$\angle A = \angle D = 40^\circ$$

And $\frac{AC}{DF} = \frac{10}{15} = \frac{2}{3}$

Also $\frac{AB}{DE} = \frac{12}{18} = \frac{2}{3}$

As $\angle A \cong \angle D$ and $\frac{AC}{DF} = \frac{AB}{DE} = \frac{2}{3}$

$\therefore \Delta ABC \sim \Delta DEF$

> If in a correspondence of two triangles ratios of two pairs of corresponding sides and one pair of corresponding angles are equal then triangles are similar.

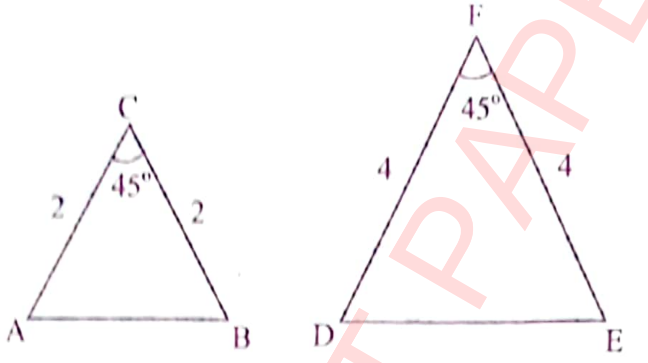
Example 4:

Check whether the following figures are similar.

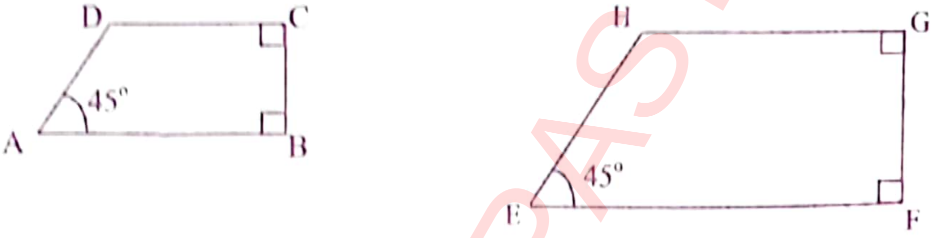
All measurements are in centimetres.

> **Check Point**
>
> In above figure, measure ($\angle C$, $\angle F$), ($\angle F$, $\angle B$) and $\frac{BC}{EF}$. What do you notice?

(i)



(ii)



Solution:

(i) $\quad \frac{AC}{DF} = \frac{BC}{EF} = \frac{1}{2}$ and $\angle C \cong \angle F$

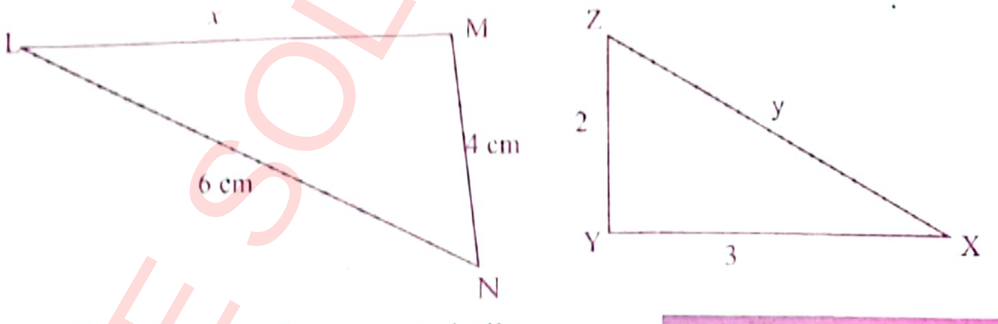
So $\Delta ABC \sim \Delta DEF$

(ii) In quadrilaterals ABCD and EFGH

$\angle A = \angle E$, $\angle B = \angle F$, $\angle C = \angle G$ and thus $\angle D = \angle H$

Therefore, both quadrilaterals are similar.

Example 5: In the figure below $\Delta LMN \sim \Delta XYZ$. Find values of x and y.



Solutions: Since the two triangles are similar.

$$\frac{x}{3} = \frac{6}{y} = \frac{4}{2}$$

$$\frac{x}{3} = \frac{4}{2} \quad \text{and} \quad \frac{6}{y} = \frac{4}{2}$$

$$x = 2 \times 3 \quad \text{and} \quad \frac{6}{y} = 2$$

> **Key Fact**
>
> Two triangles are similar if two sides of one triangle are proportional to the corresponding two sides of the second triangle and angles between the proportional sides are congruent.

$x = 6$ and $6 = 2y$

$\therefore$ $x = 6$ and $y = 3$

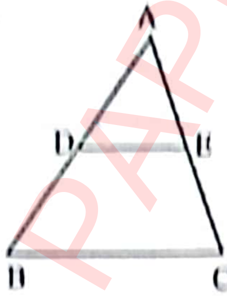
Key Fact

(i) In ΔABC, if $\overline{DE} \parallel \overline{BC}$, then $\angle B = \angle D$ and $\angle C = \angle E$.

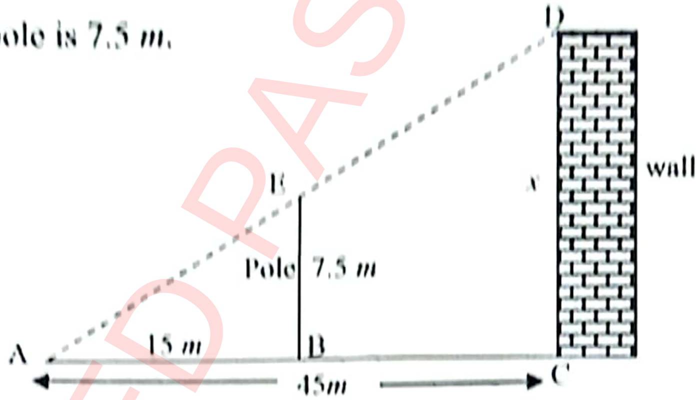
(ii) Corresponding angles in ΔABC and ΔADE are equal. So, the triangles are similar.

(iii) If ΔABC is isosceles or equilateral so is ΔADE

(iv) Also $\frac{AD}{DB} = \frac{AE}{EC}$ and $\frac{AD}{AB} = \frac{DE}{BC} = \frac{AE}{AC}$



Example 6: In the figure height of pole is 7.5 *m*. Find the height of wall.



Solution:

Let height of wall $= CD = x$

Join A to E and D.

Here $\overline{BE} \parallel \overline{CD}$ in the ΔACD

$\therefore \frac{AB}{AC} = \frac{BE}{CD}$

$\frac{15}{45} = \frac{7.5}{x}$

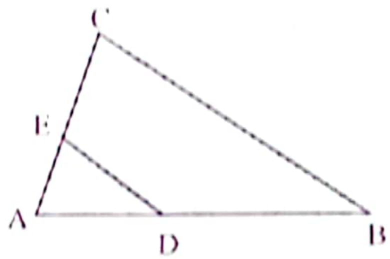
$15x = 45 \times 7.5$

$x = \frac{45 \times 7.5}{15} = 22.5$

So, height of wall $= 22.5m$

Example 7:

In ΔABC, $AB = 10\ cm$, $AC = 5\ cm$. Take D and E on $\overline{AB}$ and $\overline{AC}$ respectively such that $AD = 4\ cm$ and $AE = 2\ cm$. Is $\overline{DE} \parallel \overline{BC}$?



Solution: As $\frac{AD}{DB} = \frac{4}{6} = \frac{2}{3}$ [As $BD = AB - AD$] .... (i)

And $\frac{AE}{EC} = \frac{2}{3}$ [As $EC = AC - AE$] .....(ii)

From (i) and (ii)

$$\frac{AD}{DB} = \frac{AE}{EC},$$

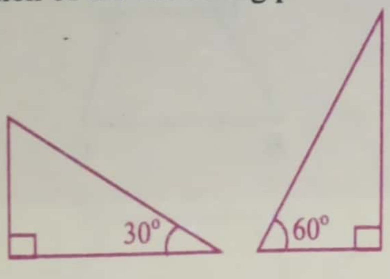
which shows that DE intersects AB and AC in the same ratio.

$$\therefore \quad \overline{DE} \parallel \overline{BC}$$

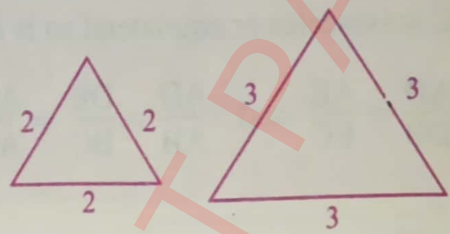
EXERCISE 9.2

1. Which of the following pairs of figures are similar?

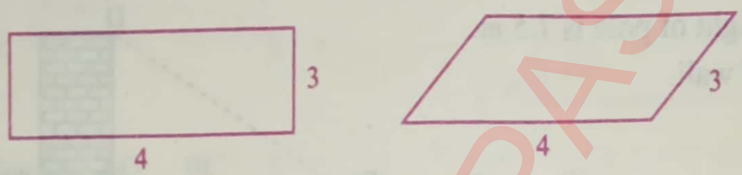
(i)



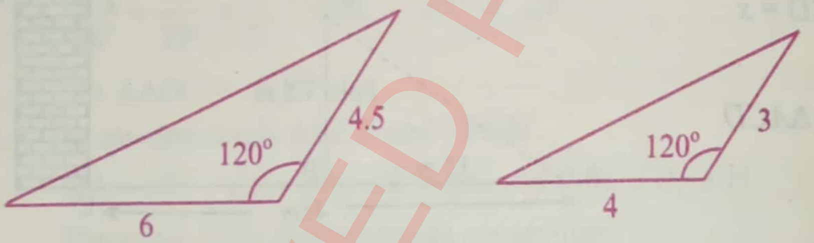
(ii)



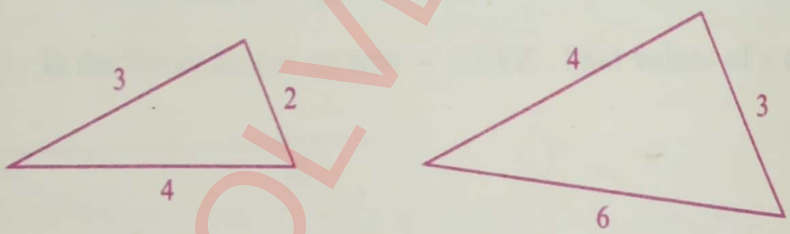
(iii)



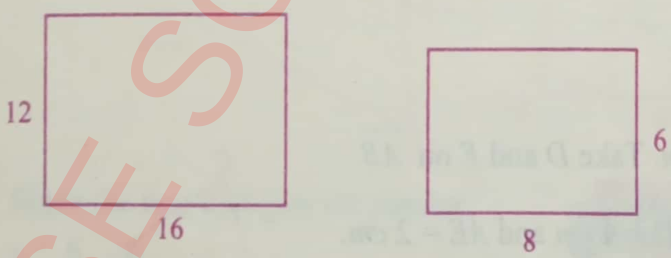
(iv)



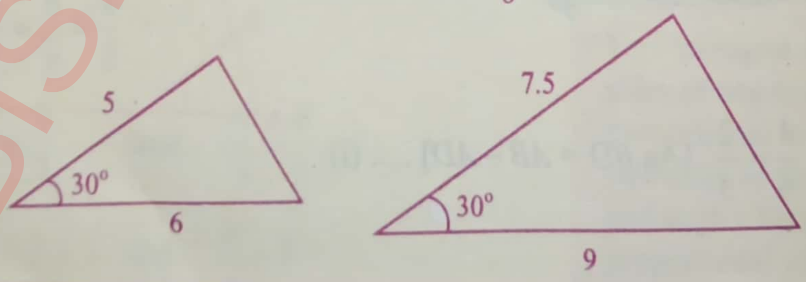
(v)



(vi)

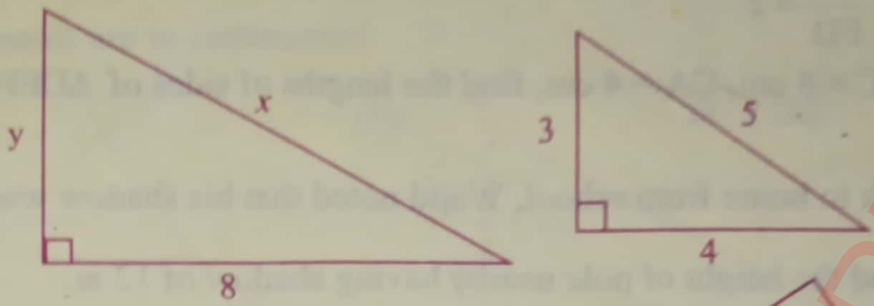


(vii)

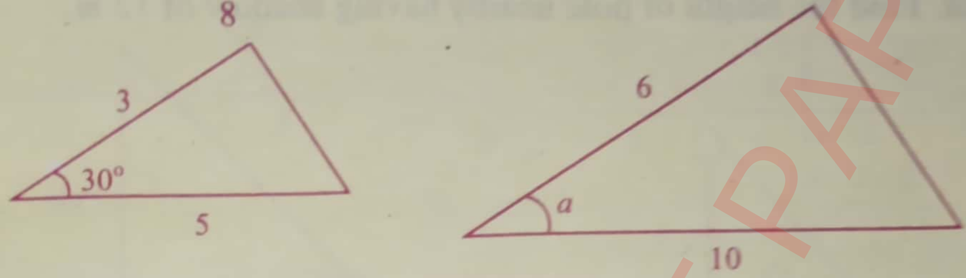


2. Find the unknown quantities in the following similar figures. All measurements are in centimetres.

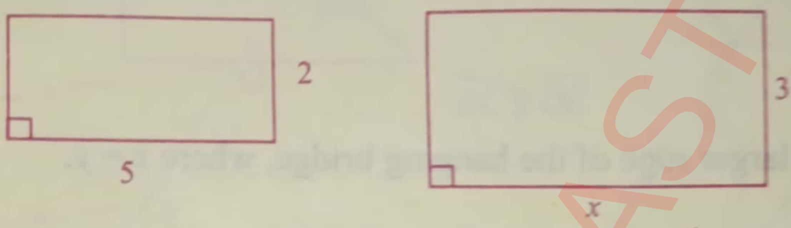
(i)



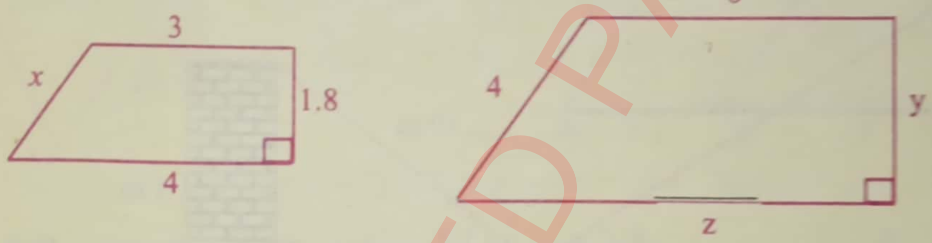
(ii)



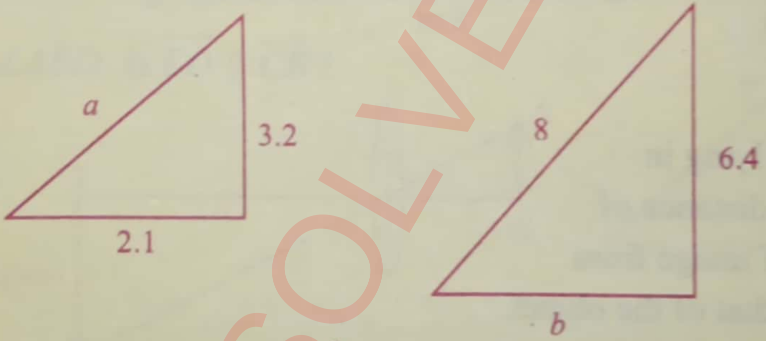
(iii)



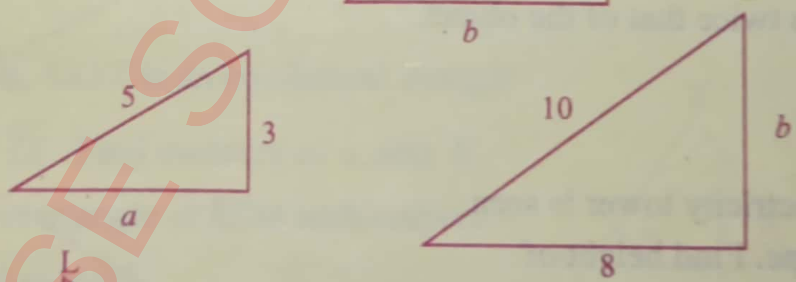
(iv)



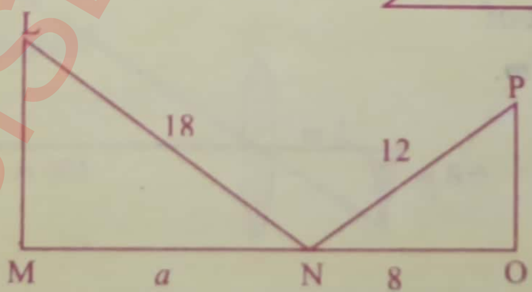
(v)



(vi)



(vii)



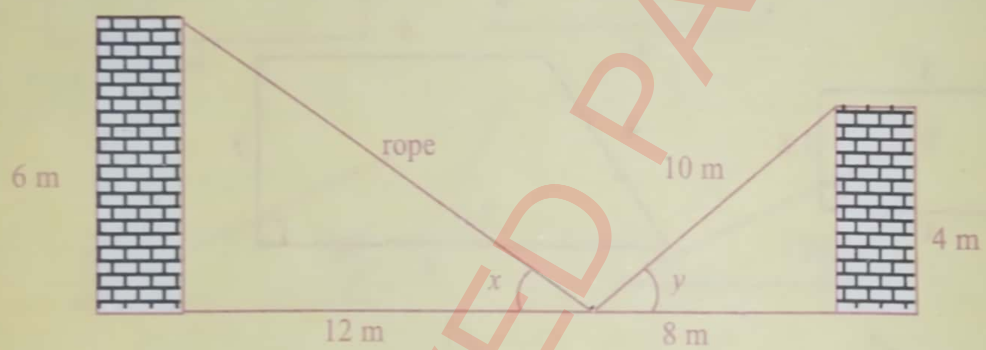
3. Two Triangles ABC and DEF are similar.

and $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD} = 2$

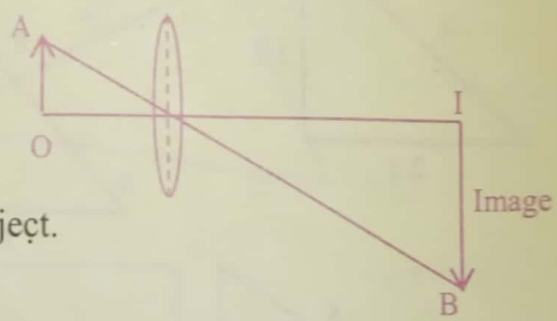
If AB = 6 cm, BC = 8 cm, CA = 4 cm, find the lengths of sides of ΔDEF.

4. While going back to home from school, Wajid noted that his shadow was $\frac{2}{3}$ of his height. Find the height of pole nearby having shadow of 12 m.

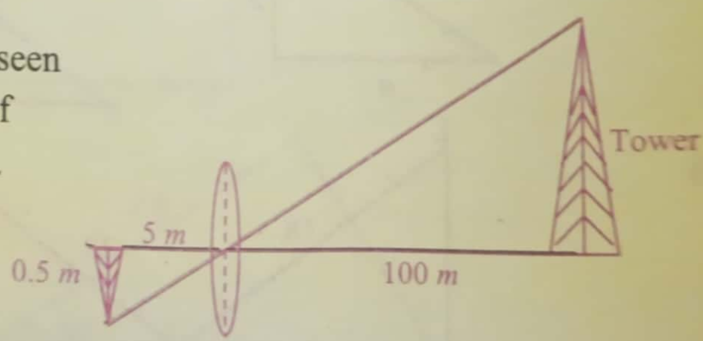
5. Find the length of the larger rope of the hanging bridge, where $x = y$.



6. In the figure, OA is object lying in front of a convex lens at a distance of 10 cm. Find the distance of image from the lens if its size is twice that of the object.

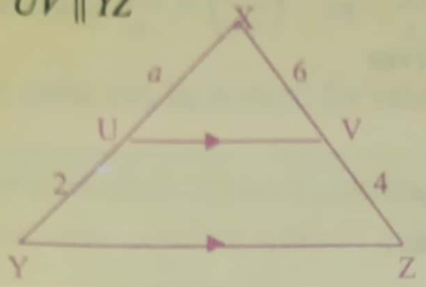


7. In the figure, an electricity tower is seen through the telescope. Find height of tower if height of its image is 0.5 m.

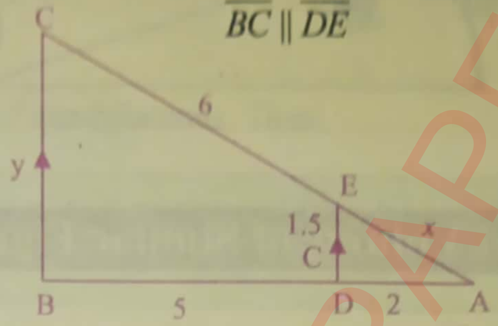


8. Find the values of unknown quantities in the following figures. All the measurements are in centimeters.

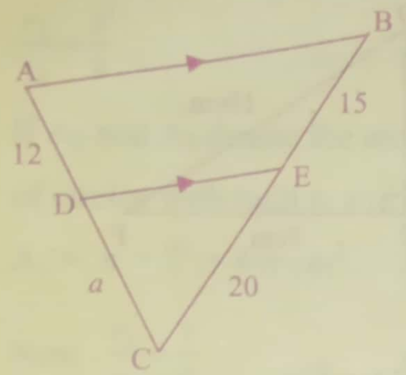
(a) $\overline{UV} \parallel \overline{YZ}$



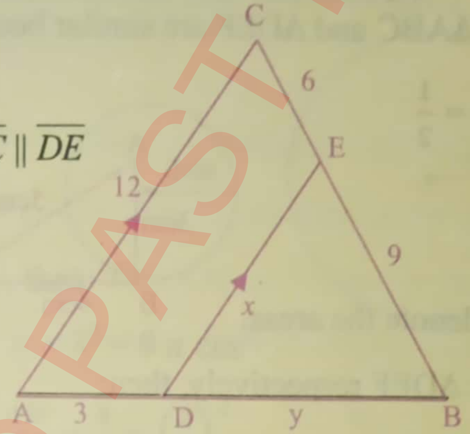
(b) $\overline{BC} \parallel \overline{DE}$



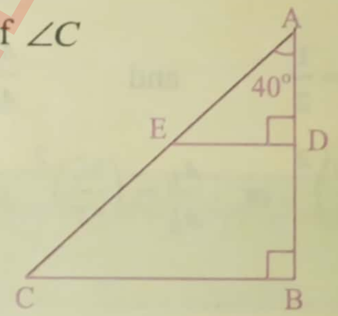
(c) $\overline{AB} \parallel \overline{DE}$



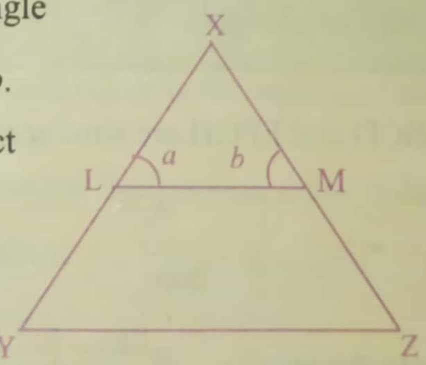
(d) $\overline{AC} \parallel \overline{DE}$



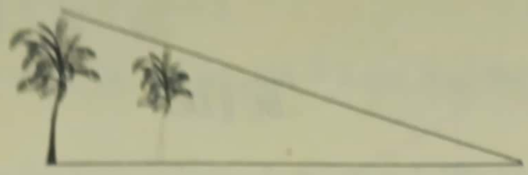
9. In the following figure, find the measure of $\angle C$ and $\angle AED$. Is $\overline{ED} \parallel \overline{CB}$?



10. In the figure, ΔXYZ is an equilateral triangle and $\overline{LM} \parallel \overline{YZ}$. Find measure of a and b. What type of triangle is XLM with respect to sides and angles?



11. Find the height of shorter tree if longer one is 12 m high from the ground level, where observer is at a distance of 25 m from the shorter tree and distance between both trees is 5 m.



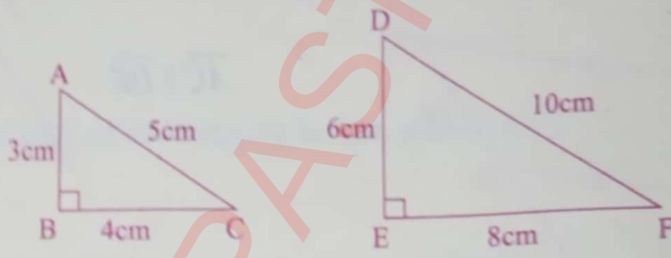
Observer

Area and Volume of Similar Figures

Area of Similar Figures

1. In the figure, ΔABC and ΔDEF are similar because:

$$\frac{AB}{DE}=\frac{BC}{EF}=\frac{CA}{FD}=\frac{1}{2}$$



If A_1 and A_2 denote the areas of ΔABC and ΔDEF respectively, then:

$A_1=\frac{1}{2}\times 4\times 3=6cm^2$ and $A_2=\frac{1}{2}\times 8\times 6=24cm^2$

Now $\frac{AB}{DE}=\frac{BC}{EF}=\frac{CA}{FD}=\frac{1}{2}$ and $\frac{A_1}{A_2}=\frac{6}{24}=\frac{1}{4}=\left(\frac{1}{2}\right)^2$

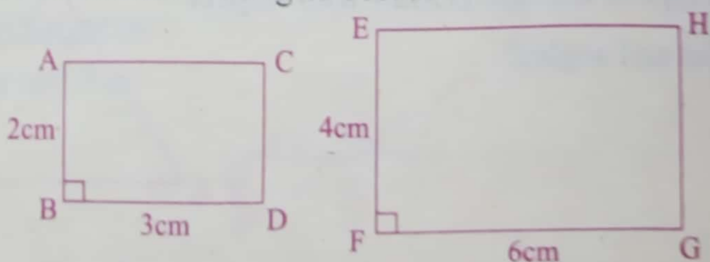
Therefore: $\frac{A_1}{A_2}=\left(\frac{AB}{DE}\right)^2$ or $\frac{A_1}{A_2}=\left(\frac{BC}{EF}\right)^2$ or $\frac{A_1}{A_2}=\left(\frac{CA}{FD}\right)^2$

We notice that:

> Ratio of areas of two triangles is equal to the square of the ratios of any two corresponding sides of triangles.

2. In the figure, ABCD and EFGH are similar rectangles because:

$$\frac{AB}{EF}=\frac{BD}{FG}=\frac{1}{2}$$



If A_1 and A_2 denote the areas of rectangles ABCD and EFGH respectively, then:

$A_1 = 2 \times 3 = 6cm^2$ and $A_2 = 4 \times 6 = 24cm^2$

Now $\frac{AB}{EF} = \frac{BD}{FG} = \frac{1}{2}$ and $\frac{A_1}{A_2} = \frac{6}{24} = \frac{1}{4} = \left(\frac{1}{2}\right)^2$

Therefore: $\frac{A_1}{A_2} = \left(\frac{AB}{EF}\right)^2$ or $\frac{A_1}{A_2} = \left(\frac{BD}{FG}\right)^2$

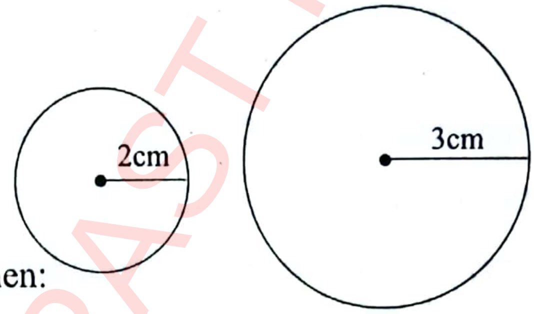
The same can be proved for other types of quadrilaterals. Thus:

> Ratio of areas of two quadrilaterals is equal to the square of the ratios of any two corresponding sides of quadrilaterals.

3. Figure shows two similar circles with radii 2cm and 3cm respectively.

If $r_1 = 2cm$ and $r_2 = 3cm$, then:

$\frac{r_1}{r_2} = \frac{2}{3}$



If A_1 and A_2 denote the areas of circles with radii r_1 and r_2 respectively, then:

$A_1 = \pi \times 2^2 = 4\pi\ cm^2$ and $A_2 = \pi \times 3^2 = 9\pi\ cm^2$

Now $\frac{r_1}{r_2} = \frac{2}{3}$ and $\frac{A_1}{A_2} = \frac{4\pi}{9\pi} = \frac{4}{9} = \left(\frac{2}{3}\right)^2$

Therefore: $\frac{A_1}{A_2} = \left(\frac{r_1}{r_2}\right)^2$

Thus:

> Ratio of areas of two circles is equal to the square of the ratios of radii of both circles.

Key Fact

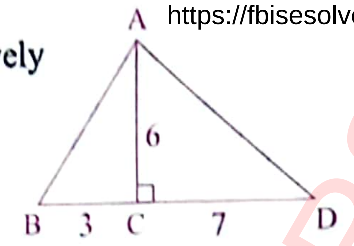
The ratio of areas of two similar figures is equal to the square of ratios of any two corresponding lengths of the figures.

If A_1 and A_2 denote the areas of two similar figures and, a and b denote the corresponding lengths of the figures, then:

$$\frac{A_1}{A_2} = \left(\frac{a}{b}\right)^2 = k^2$$

Example 8:

Triangles ABC and ADC with bases 3cm and 7cm respectively have common height 6 cm.
Prove that the ratio of their areas is equal to ratio of bases of both triangles.



Solution:

Area of $\Delta ABC = \frac{1}{2} \times 3 \times 6 = 9\text{cm}^2$

Area of $\Delta ADC = \frac{1}{2} \times 7 \times 6 = 21\text{cm}^2$

$$\frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta ADC} = \frac{9}{21} = \frac{3}{7} = \frac{\text{Base of } \Delta ABC}{\text{Base of } \Delta ADC}$$

Key Fact

Ratio of areas of two triangles with common height is equal to the ratio of bases of the two triangles.

$$\frac{A_1}{A_2} = \frac{b_1}{b_2} = k$$

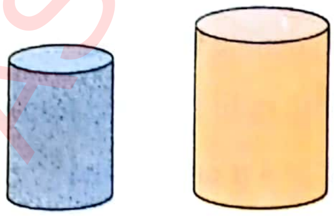
Example 9:

The ratio of the areas of the bases of two similar cylinders is 9 : 4.

The area of the base of the smaller cylinder is 240cm^2.

(i) What is the area of base of larger cylinder?

(ii) Write the ratio of heights of both cylinders.



Solution:

(i) If A_1 and A_2 are areas of bases of larger and smaller cylinders respectively, then:

$$\frac{A_1}{A_2} = \frac{9}{4} \quad \Rightarrow \quad \frac{A_1}{240} = \frac{9}{4}$$

$$\Rightarrow \quad A_1 = \frac{9}{4} \times 240 = 540\text{cm}^2$$

(ii) Again, if h_1 and h_2 are heights of larger and smaller cylinders respectively, then:

$$\left(\frac{h_1}{h_2}\right)^2 = \frac{A_1}{A_2} \quad \Rightarrow \quad \left(\frac{h_1}{h_2}\right)^2 = \frac{540}{240} = \frac{9}{4}$$

$$\Rightarrow \quad \frac{h_1}{h_2} = \frac{3}{2}$$

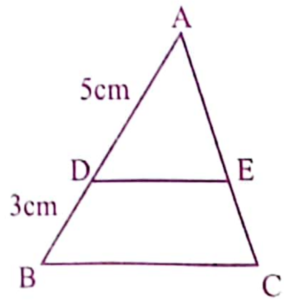
Example 10:

In the figure, $\overline{BC} \parallel \overline{DE}$. Find:

(i) $\frac{DE}{BC}$ (ii) $\frac{\text{Area of } \Delta ADE}{\text{Area of } \Delta ABC}$

(iii) find the area of ΔADE if area of ΔABC is 256cm^2.

(iv) area of trapezium DBCE.



Solution:

(i) $\frac{DE}{AB} = \frac{AD}{AB} = \frac{5}{8}$

(ii) $\frac{\text{Area of } \Delta ADE}{\text{Area of } \Delta ABC} = \left(\frac{5}{8}\right)^2 = \frac{25}{64}$

(iii) $\frac{\text{Area of } \Delta ADE}{\text{Area of } \Delta ABC} = \frac{25}{64} \Rightarrow \text{Area of } \Delta ADE = \frac{25}{64} \times \text{Area of } \Delta ABC$

$\Rightarrow \text{Area of } \Delta ADE = \frac{25}{64} \times 256 \text{ cm}^2 = 100 \text{ cm}^2$

(iv) Area of trapezium DBCE = Area of ΔABC – Area of ΔABC

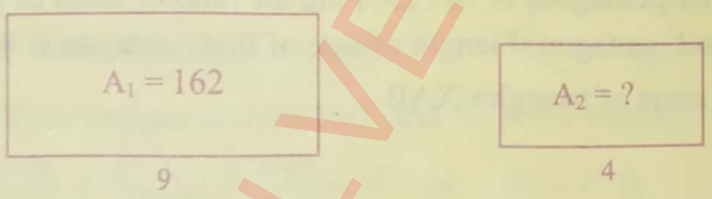
$= 256 - 100 = 156 \text{ cm}^2$

EXERCISE 9.3

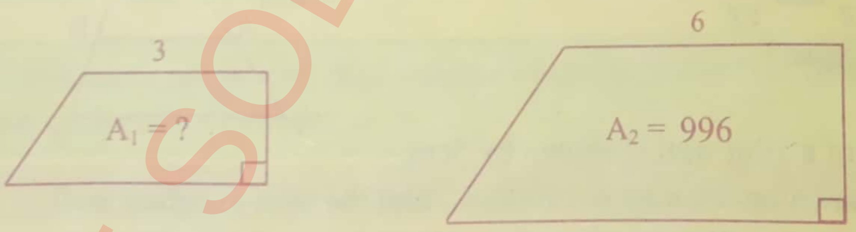
1. Following pairs of shapes are similar. Find unknown area in each case. All measurements are in cm and cm^2.

(i)

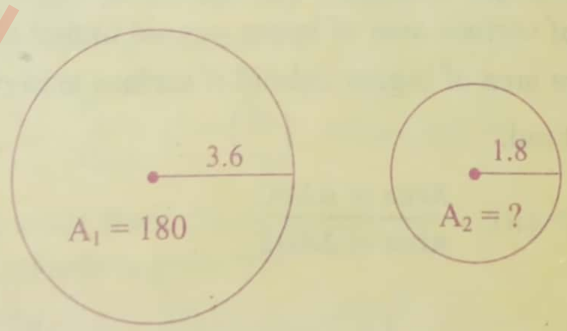
(ii)



(iii)

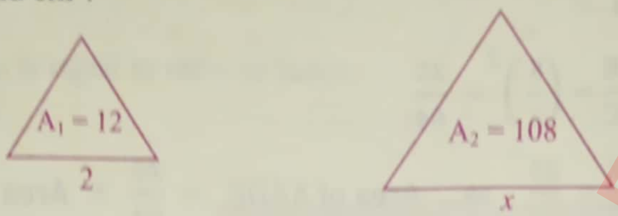


(iv)

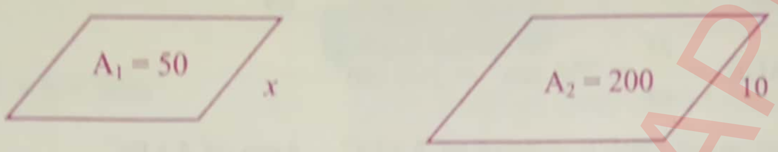


2. Following pairs of shapes are similar. Find unknown length x in each case. All measurements are in cm and cm^2.

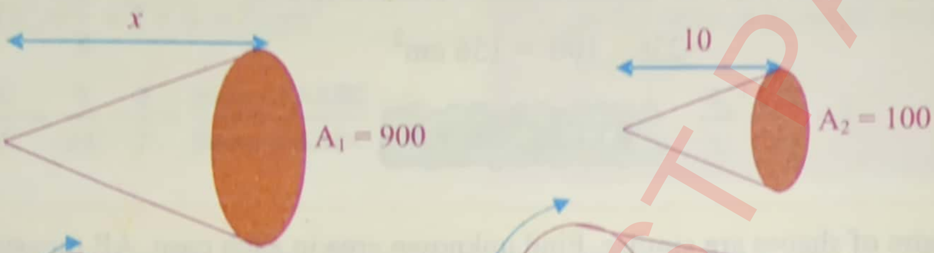
(i)



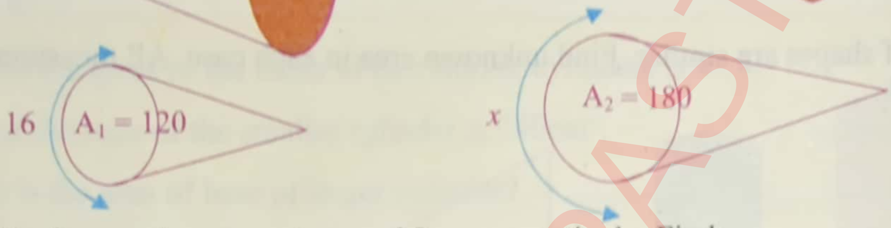
(ii)



(iii)



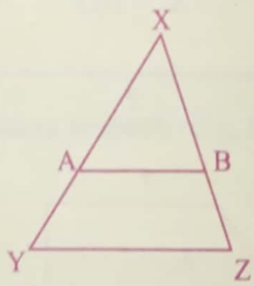
(iv)



3. Radii of two spheres are 6cm and 8cm respectively. Find:
   (i) the ratio of areas of both spheres.
   (ii) area of larger sphere if area of smaller sphere is $360cm^2$.
   (iii) area of smaller sphere if area of larger sphere is $1600cm^2$.

4. Ratio of areas of two regular pentagons is 16 : 25. Find the ratio of sides of pentagons. Also find length of side of second pentagon if length of side of first pentagon is 8cm.

5. In the figure, $\overline{AB} \parallel \overline{YZ}$. If areas of triangles XAB and XYZ are in the ratio 25 : 36, find $\frac{XB}{XZ}$ and $\frac{AB}{YZ}$. Are the ratios equal?

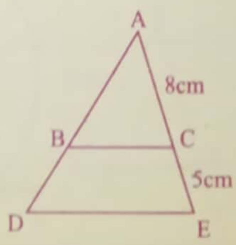


6. In a map, length of a 10m wall is shown by 5cm. If area of wall shown on the map is $1400cm^2$, find the area of actual wall.

7. Two cuboids are similar. Height of smaller cuboid is one-third of bigger one.
   (i) Find the ratio of surface area of larger cuboid to that of smaller one.
   (ii) Find the surface area of bigger cuboid if surface area of smaller cuboid is $350cm^2$.

8. In the figure, $\overline{BC} \parallel \overline{DE}$. Find:

   (i) $\frac{BC}{DE}$ and $\frac{AB}{AD}$ (ii) $\frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta ADE}$



(iii) the area of ΔABC if area of ΔADE is $507 cm^2$.

(iv) area of quadrilateral BDEC.

What type of the quadrilateral is?

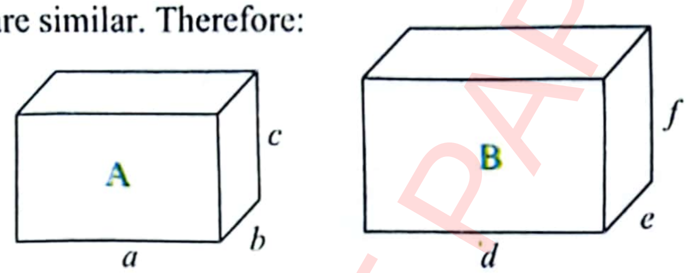
Volume of Similar Solids

Two solids are said to be similar if they have same shape. The ratio of corresponding lengths (sides etc.) of two similar solids is constant called the scale factor of the solids.

1. In the figure two cuboids are similar. Therefore:

$$\frac{a}{d} = \frac{b}{e} = \frac{c}{f} = k$$

$$\Rightarrow a = dk, b = ek, c = fk$$



Now if V_1 is volume of cuboid A and V_2 is the volume of cuboid B, then

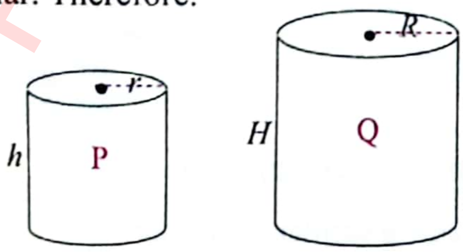
$$\frac{V_1}{V_2} = \frac{abc}{def} = \frac{dkekfk}{def} = k^3 \quad \Rightarrow \quad V_1 = k^3 V_2$$

> If two cuboids are similar then volume of one cuboid is k^3 times the volume of the other cuboid.

2. In the figure two cylinders P and Q are similar. Therefore:

$$\frac{r}{R} = \frac{h}{H} = k$$

$$\Rightarrow r = Rk \text{ and } h = Hk$$



Now if V_1 is volume of cylinder P and V_2 is the volume of cylinder Q, then

$$\frac{V_1}{V_2} = \frac{\pi r^2 h}{\pi R^2 H} = \frac{r^2 h}{R^2 H} = \frac{R^2 k^2 Hk}{R^2 H} = k^3 \quad \Rightarrow \quad V_1 = k^3 V_2$$

> If two cylinders are similar then volume of one cylinder is k^3 times the volume of the other cylinder.

Key Fact

If l_1 and l_2 are any two corresponding lengths of two similar figures then the ratio of corresponding volumes is:

$$\frac{V_1}{V_2} = k^3 = \left(\frac{l_1}{l_2}\right)^3 \text{ where } k = \frac{l_1}{l_2}$$

If both solids are made from the same material or have the same density, the ratio of their masses is given by:

$$\frac{m_1}{m_2} = k^3 = \left(\frac{l_1}{l_2}\right)^3$$

Example 11: Check whether the prisms are similar or not?

Solution:

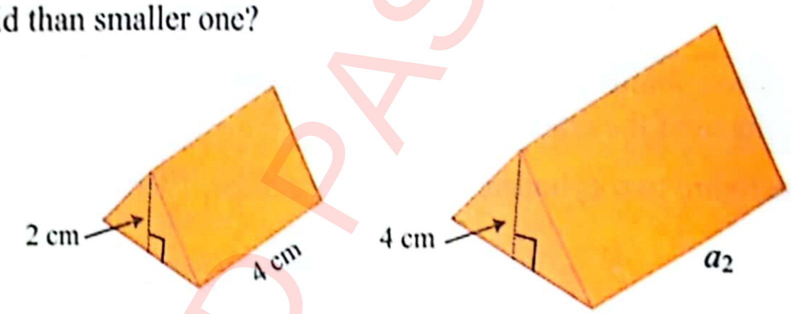
Let $a = 4$, $b = 3$, $c = 5$ and $d = 6$, $e = 4.5$, $f = 7.5$.

Then: $\frac{a}{d} = \frac{4}{6} = \frac{2}{3}$, $\frac{b}{e} = \frac{3}{4.5} = \frac{30}{45} = \frac{2}{3}$, $\frac{c}{f} = \frac{5}{7.5} = \frac{50}{75} = \frac{2}{3}$

Therefore, the two prisms are similar.

Example 12:

Find a_2 it the following solids are similar. Also find the ratio of volumes of both. **How many times** is the volume of larger solid than smaller one?



Solution:

Let $h_1 = 2$cm, $h_2 = 4$cm, $a_1 = 4$cm, $a_2 = ?$

As the figures are similar, therefore:

$$\frac{h_1}{h_2} = \frac{a_1}{a_2} \Rightarrow a_2 = \frac{a_1 h_2}{h_1} = \frac{4 \times 4}{2} = 8\text{cm}$$

Now, $\frac{V_1}{V_2} = \left(\frac{h_1}{h_2}\right)^3 = \left(\frac{2}{4}\right)^3 = \frac{1}{8}$

$\Rightarrow \quad V_2 = 8V_1$

Hence, volume of larger solid is 8 times the volume of smaller one.

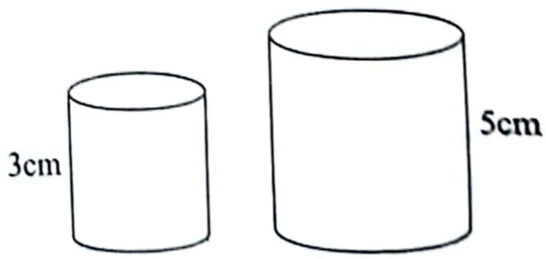
Check Point

If two solids are similar, what is the ratio of their surface areas, and what is the ratio of their volumes?

Example 13:

In the figure two geometrically similar cylinders are shown. Find:

(i) the ratio of volume of smaller cylinder to larger cylinder.



(ii) curved surface area of smaller cylinder if that of larger one is $250cm^2$.

(iii) volume of larger cylinder if volume of smaller one is $162cm^3$.

Solution:

(i) If V_1 and V_2 are volumes of smaller and larger cylinders respectively, then:

$$\frac{V_1}{V_2} = \left(\frac{3}{5}\right)^3 \Rightarrow \frac{V_1}{V_2} = \frac{27}{125}$$

(ii) If A_1 and A_2 are areas of smaller and larger cylinders respectively, then:

$$\frac{A_1}{A_2} = \left(\frac{3}{5}\right)^2 \Rightarrow \frac{A_1}{250} = \frac{9}{25}$$

$$\Rightarrow A_1 = \frac{9}{25} \times 250 = 90cm^2$$

(iii) $$\frac{162}{V_2} = \left(\frac{3}{5}\right)^3 \Rightarrow \frac{162}{V_2} = \frac{27}{125}$$

$$\Rightarrow 27 \times V_2 = 162 \times 125 \Rightarrow V_2 = \frac{162 \times 125}{27} = 750cm^3$$

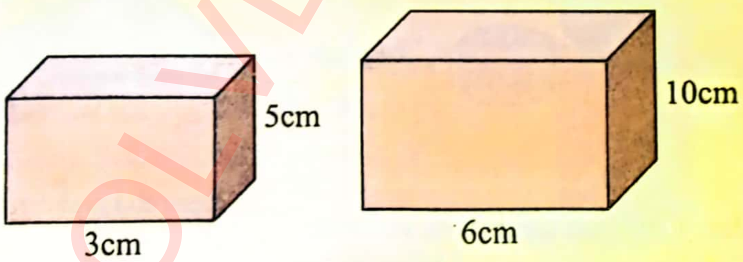
Key Fact

The ratio of volumes of two similar solids is equal to the cube of the ratio of any two corresponding lengths of the two solids.

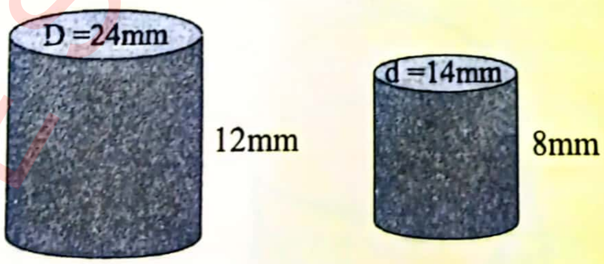
EXERCISE 9.4

1. Determine whether the solids are similar or not.

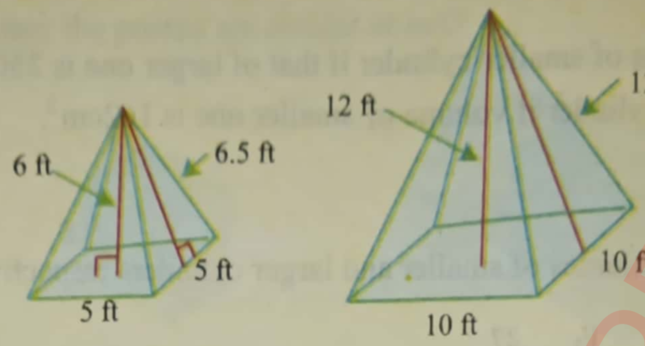
(i)



(ii)



(iii)



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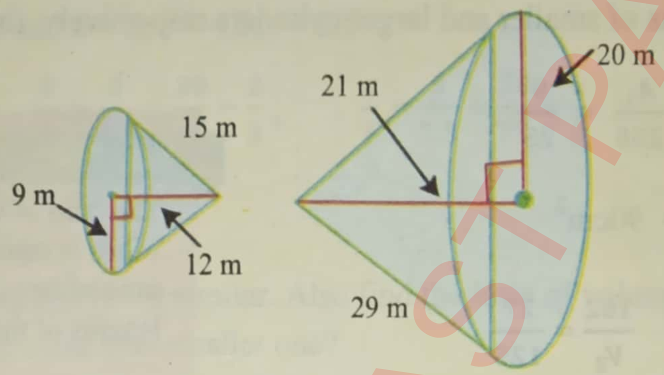
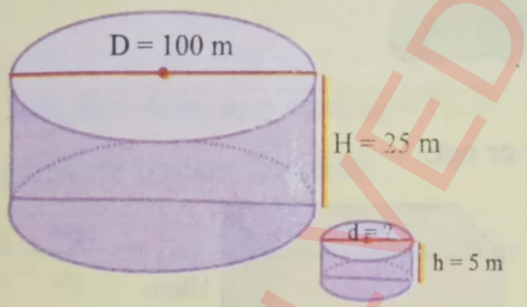
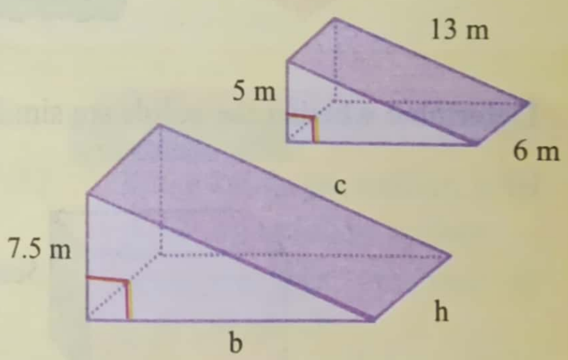


2. Solids are similar. Find the values of unknowns. Also find the ratios of volume of solids.

(i)

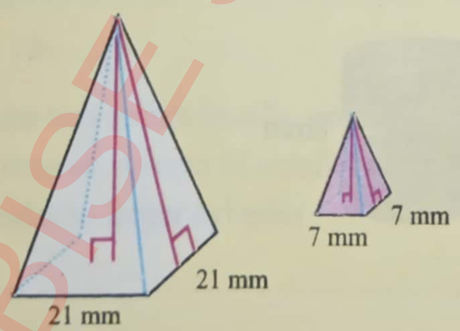


(ii)



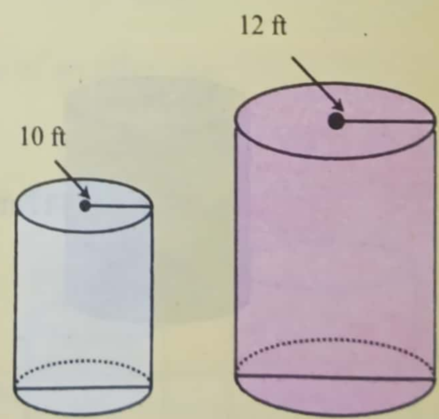
3. Solids are similar. Find the unknown volume.

(i)



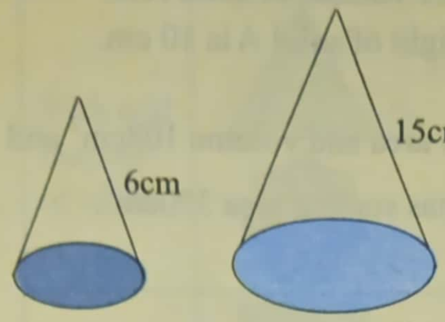
$V_1 = 9000\text{mm}^3$ $\quad V_2 = ?$

(ii)



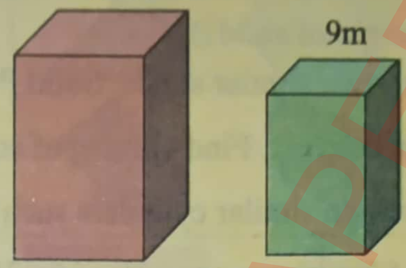
$V_1 = ?$ $\quad V_2 = 360\text{ft}^3$

(iii)



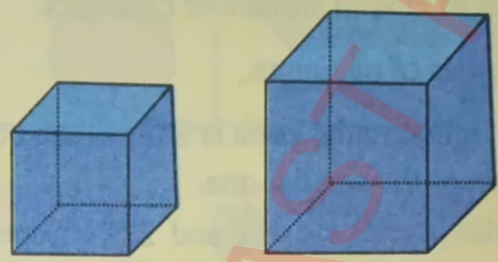
$V_1 = ?$ $V_2 = 460cm^3$

(iv)



$V_1 = 288m^3$ $V_2 = ?$

4. Find the ratio of scale factors of the following pairs of similar solids.



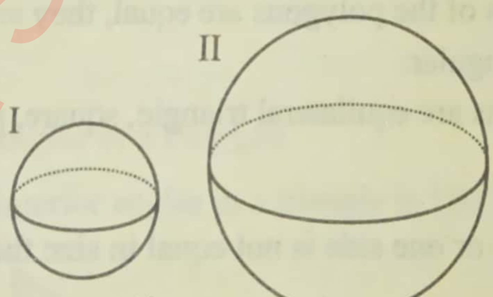
$V = 512\ m^3$ $V = 1728\ m^3$

5. Two swimming pools are similar with a scale factor of 4 : 5. The amount of chlorine mixture to be added is proportional to the volume of water in the pool. If three cups of chlorine mixture are needed for the smaller pool, how much of the chlorine mixture is needed for the larger pool?

6. A model bus is built with a scale of 1 : 10. The model bus has a volume of 30 m^3. What is the volume of the actual bus?

7. Solid A shown below is similar to solid B (not shown) with a scale factor of 2 : 3. Find the surface area and volume of solid B if surface area and volume of solid A are $130\pi\ cm^2$ and $280\pi\ cm^3$ respectively.

8. Solid I is similar to Solid II. Find the scale factor of solid I to solid II.



$V = 8\pi ft^3$ $V = 125\pi ft^3$

9. Solid A and solid B are mathematically similar. The volume of solid A is $32\ cm^3$. The volume of solid B is $108\ cm^3$. The height of solid A is 10 cm. Find the height of solid B.

10. P and Q are two similar solids. Solid P has surface area and volume $108cm^2$ and $135cm^3$ respectively. Find volume of solid Q if Q has surface area $300cm^2$.

11. X and Y are two similar cylinders such that:

    base area of X : base area of Y = 16 : 25. Find:

    (i) ratio of heights of both cylinders.

    (ii) Ratio of areas of curved surface of cylinders.

    (iii) Ratio of volumes of cylinders.

12. The volume of one right circular cone is 8 times the other one. If the radius of larger cone is 12cm, find the radius of the smaller one.

13. Masses of two similar objects are 8 kg and 27kg respectively. If the height of first object is 2m, what is the height of second object?

Properties of Regular Polygons

Polygon

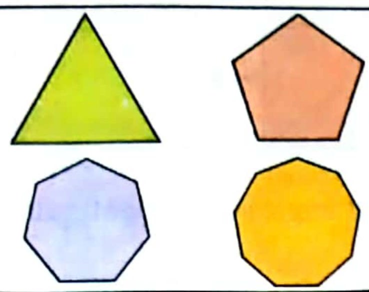
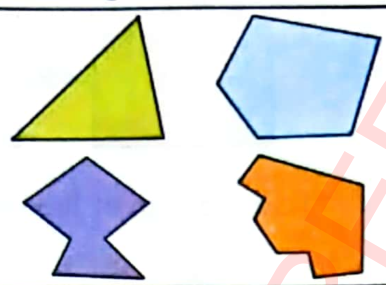
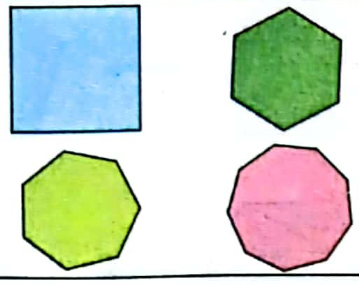
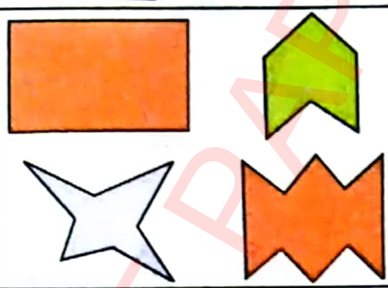
'Polygon' is the Greek word where 'poly' means 'many' and 'gon' means 'angles'.
A polygon is a two-dimensional convex figure that has a finite number of sides (at least three sides). The sides (edges) of a polygon are made of straight line segments connected end to end to form a closed shape.
The point where two line segments meet is called vertex of polygon.

Regular Polygon

If all the sides and interior angles of the polygons are equal, they are known as regular polygons.
Regular polygons are also equiangular.
The examples of regular polygons are equilateral triangle, square, pentagon, hexagon etc.

Irregular Polygon

If in a polygon at least one angle or one side is not equal in size then the polygon is called irregular.

	Regular Polygons	Irregular Polygons
With odd number of sides		
With even number of sides		

Key Fact

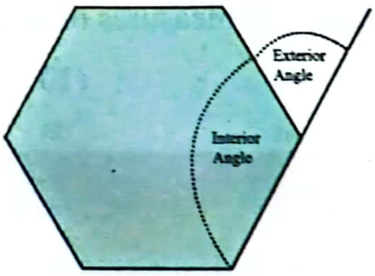
- A polygon with 5 sides is called pentagon.
- A polygon with 6 sides is called hexagon.
- A polygon with 7 sides is called heptagon.
- A polygon with 8 sides is called octagon.
- A polygon with 9 sides is called nonagon.
- A polygon with 10 sides is called decagon.

Check Point

can you search for the name of 15-sided polygon?

Interior Angle of Polygon

Interior angles are the angles that are inside the polygon formed by two adjacent sides.

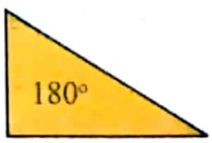


Exterior Angle of Polygon

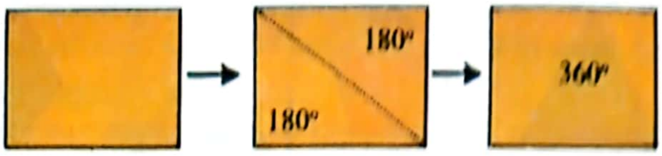
Exterior angle is the angle formed by any side of the polygon and the extension of its adjacent side.

Sum of Measures of Interior Angles of a Polygon

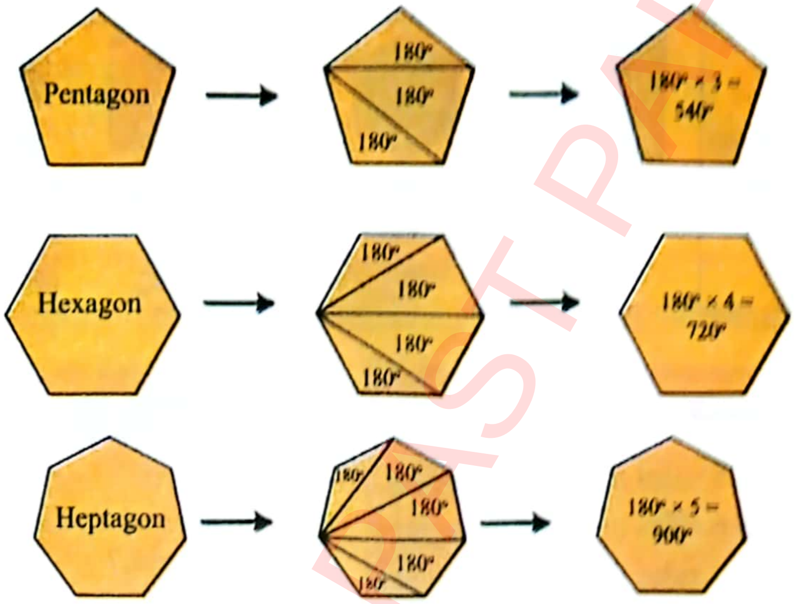
(i) Sum of measures of interior angles in a triangle is 180°.



(ii) Sum of measures of interior angles in a quadrilateral is 360°.



In this way, we can find the sum of the interior angles of any polygon by splitting it into triangles.



From the above discussion, it can be concluded that:
The sum of measures of all the interior angles of a polygon having n sides is:

$$(n-2) \times 180^\circ$$

For example, if a polygon has 10 sides, then $n = 10$.
Therefore, sum of measures of interior angles is:

$$(10-2) \times 180^\circ = 1440^\circ$$

Check Point

What is sum of measures of interior angles of polygons having?

(i) 9 sides (ii) 17 sides

Interior Angle of a Regular Polygon

The measure of each interior angle of n-sided regular polygon is given by the formula:

$$\frac{(n-2) \times 180^\circ}{n}$$

Example 14:

Find interior angle of a regular nonagon.

Check Point

Find the measure of interior angle of a regular polygon having 16 sides.

Solution:

A nonagon has 9 sides, therefore $n = 9$

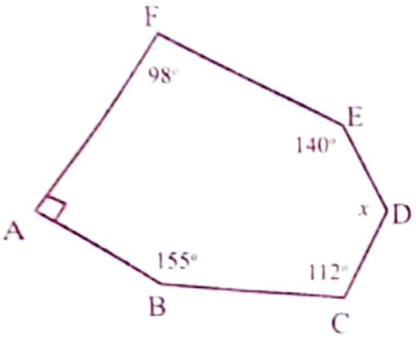
If θ is an interior angle of the nonagon then:

$$\theta = \frac{(9-2)\times 180^\circ}{9} = 140^\circ$$

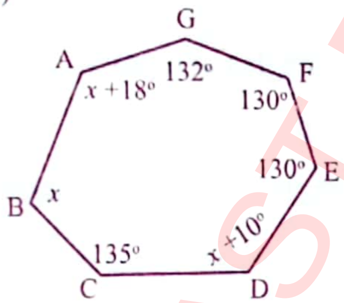
Example 15:

Find the value of x in the following polygons.

(i)



(ii)



Solution:

(i) The given polygon is hexagon and sum of measures of its interior angles is 720°.

$\therefore \quad 90^\circ + 155^\circ + 112^\circ + x + 140^\circ + 98^\circ = 720^\circ$

$\Rightarrow \quad x + 595 = 720$

$\Rightarrow \quad x = 720 - 595 = 125$

(ii) The given polygon is heptagon and sum of measures of its interior angles is 900°.

$\therefore \quad (x + 18^\circ) + x + 138^\circ + (x + 10^\circ) + 130^\circ + 130^\circ + 132^\circ = 900^\circ$

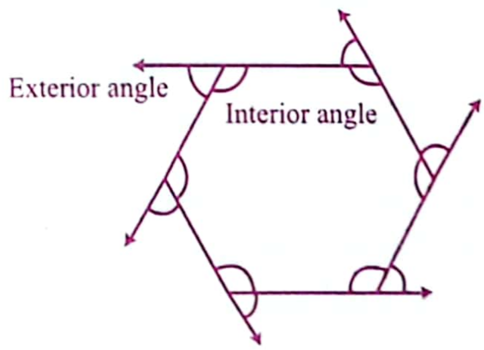
$\Rightarrow \quad 3x + 558^\circ = 900^\circ$

$\Rightarrow \quad 3x = 900^\circ - 558^\circ = 342^\circ$

$\Rightarrow \quad x = \frac{342^\circ}{3} = 114^\circ$

Sum of Measures of Exterior Angles of a Polygon

Figure shows a regular hexagon.



As interior angle of a regular hexagon is 120° and sum of measures of an interior and exterior angles is 180°, therefore:

Interior angle + Exterior angle = 180°

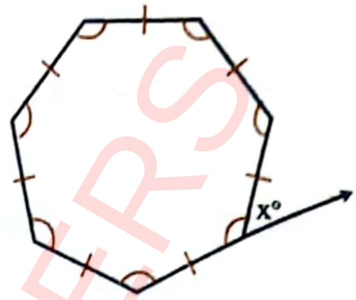
120° + Exterior angle = 180°

$\Rightarrow$ Exterior angle = $180^\circ - 120^\circ = 60^\circ$

Now, hexagon has 6 sides, therefore:

Sum of measures of exterior angles of regular hexagon = $6 \times 60^\circ$

$= 360^\circ$



For a regular polygon having n sides:

exterior angle + interior adjacent angle = 180°

Sum of all exterior angles + Sum of all interior angles

$$= n \times 180^\circ$$

So, sum of all exterior angles

$$= n \times 180^\circ - \text{Sum of all interior angles}$$

$$\text{Sum of all exterior angles} = n \times 180^\circ - (n-2) \times 180^\circ$$

$$= n \times 180^\circ - n \times 180^\circ + 2 \times 180^\circ$$

$$= 180^\circ n - 180^\circ n + 360^\circ = 360^\circ$$

∴ Sum of measures of exterior angles of a polygon is 360°.

Exterior Angle of a Regular Polygon

The measure of each exterior angle of an n-sided regular polygon is:

$$\frac{360^\circ}{n}$$

Example 16:

Exterior angle of a regular polygon is 120°. Identify the name of polygon.

Solution:

Measure of exterior angles = 360°

Measure of one exterior angle = 120°

Number of exterior angles $= \frac{360^\circ}{120^\circ} = 3$

Since the polygon has 3 exterior angles therefore, it has 3 sides.

Hence it is an equilateral triangle.

Key Fact

- All the sides and angles of a regular polygon are equal.
- The perimeter of a regular polygon with n sides is equal to the n times of a side measure.
- The number of diagonals in a polygon with n sides are $n(n-3)/2$.
- The number of triangles formed by joining the diagonals from one corner of a polygon are $n-2$.
- Number of triangles created inside a polygon is 2 less than number of sides of polygon.
- Interior and exterior angles add up to 180°.

Example 17:

In a certain polygon, the sum of the measures of all the interior angles is equal to twice that of the exterior angles. What is the name of that polygon?

Solution:

Given that

Sum of interior angles = 2 × the sum of exterior angles

If n is number of sides of polygon, then:

$$(n-2) \times 180^\circ = 2 \times 360^\circ$$

$$\Rightarrow \; n-2 = \frac{2 \times 360^\circ}{180^\circ} \; \Rightarrow \; n-2 = 4 \; \Rightarrow \; n = 6$$

Hence the polygon is hexagon.

Example 18:

The measure of the exterior angles of a polygon are $(x+4)^\circ$, $(3x-4)^\circ$, $(7x-3)^\circ$, $(2x+3)^\circ$, $(8x-1)^\circ$ and $(9x+1)^\circ$.

(i) Identify polygon. (ii) Find x. (iii) Find the measure of each angle.

Solution:

(i) Since there are 6 exterior angles in the polygon, therefore polygon is hexagon.

(ii) As the sum f measures of exterior angles in a polygon is 360°, therefore:

$$(x+4)^\circ + (3x-4)^\circ + (7x-3)^\circ + (2x+3)^\circ + (8x-1)^\circ + (9x+1)^\circ = 360^\circ$$

$$\Rightarrow \; x + 4^\circ + 3x - 4^\circ + 7x - 3^\circ + 2x + 3^\circ + 8x - 1^\circ + 9x + 1^\circ = 360^\circ$$

$$\Rightarrow \; 30x = 360^\circ \quad \Rightarrow \; x = 12^\circ$$

(iii) Substituting the value of x in above expressions of angles, we get:

$(12+4)^\circ$, $(3 \times 12 - 4)^\circ$, $(7 \times 12 - 3)^\circ$, $(2 \times 12 + 3)^\circ$, $(8 \times 12 - 1)^\circ$, $(9 \times 12 + 1)^\circ$

Or 16°, 32°, 81°, 27°, 95°, 109°

EXERCISE 9.5

1. Find the number of sides of a regular polygon if each of its exterior angle is:

   (a) 45° (b) 60° (c) 120° (d) 40°

2. Draw a regular pentagon whose exterior angles are p, q, r, s, t and each interior angle is k. What is the measure of:

   (a) each interior angle (b) each exterior angle

3. Each interior angle of a polygon is five times the exterior angle of the polygon. Find the number of sides.
4. Find the minimum interior angles and maximum exterior angles possible in a regular polygon. Give reasons to support your answer.
5. Find the exterior angle of a regular polygon of:

   (a) 5 sides (b) 9 sides (c) 15 sides (d) 20sides

6. The ratio between an exterior angle and the interior angle of a regular polygon is 1 : 2. Find:

   (a) the measure of each exterior angle.
   (b) the measure of each interior angle.
   (c) the number of sides in the polygon.

7. Is it possible to have a regular polygon each of whose exterior angle is 50°? Give reason to support your answer.
8. Name the polygon whose sum of interior angles is equal to the sum of its exterior angles.
9. The sum of all the interior angles of a regular polygon is four times the sum of its exterior angles. Identify the polygon.
10. An exterior angle of a regular polygon is 12°. What is the sum of all the interior angles?
11. Prove that each interior angle and its corresponding exterior angle in any polygon are supplementary.
12. Find the number of sides in a regular polygon when the measure of each exterior angle is 72°.
13. The exteriors angles of a pentagon are $(y + 5)°$, $(2y + 3)°$, $(3y + 2)°$, $(4y + 1)°$ and $(5y + 4)°$ respectively. Find the measure of each angle.
14. A convex polygon has 14 diagonals. Find the number of sides of the polygon.
15. Find the sum of all the interior angles of a polygon having 13 sides.
16. The sum of all the interior angles of a polygon is 2880°. How many sides does the polygon have?

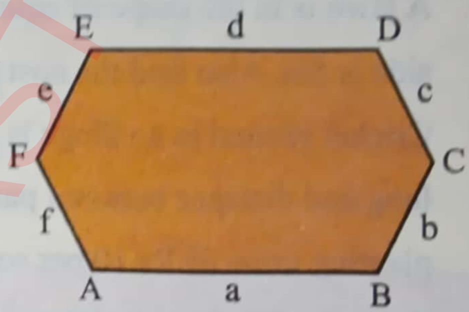
Real Life Problems Involving Regular Polygons, Triangles and Parallelograms

The variety of polygons are commonly used in the modern constructions. Because of its reasonably strong design, the triangle is commonly used in construction. The usage of the polygons minimizes the number of resources used to construct a structure like lowering costs and increasing profits in a corporate setting. The rectangle is another polygon which is used in a variety of applications. For example, most of televisions are rectangular to make watching easier and more enjoyable. Photo frames and phone screens are in the same boat.

Perimeter of Polygon

Perimeter of any polygon is sum of measures of all sides.

Perimeter of pentagon shown in adjoining figure, is:



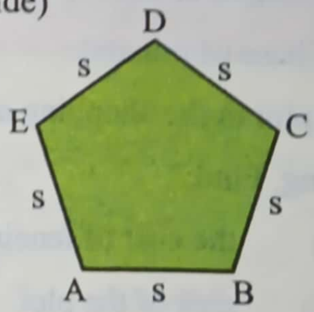
$$P = a + b + c + d + e + f$$

Perimeter of Regular Polygon

Perimeter of regular polygon = (number of sides) × (length of one side)

$$P = n \times s$$

Perimeter of a regular pentagon shown in adjoining figure, is:

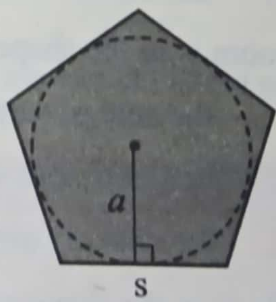


$$P = s + s + s + s + s = 5s$$

Area of Regular Polygon

Area of a regular polygon is defined as:

$$A = \frac{1}{2}(P)(a)$$



Where 'P' is perimeter of regular polygon and 'a' is called apothem. Substituting the value of 'P', we have:

$$A = \frac{1}{2}(n\,s)(a) = \frac{n}{2} \times (sa)$$

Where 's' denotes the length of side.

$$\therefore\ A = \frac{1}{2}(P)(a) = \frac{n}{2} \times (sa)$$

Example 19:

Basement of a water tank is in the shape of regular pentagon having the side length 8 feet. Its apothem is 7 feet. Find the perimeter and area of basement of tank.

Solution:

Number of sides of basement = 5

Side length of basement = s = 8 ft

Apothem of basement = 7 ft

Perimeter of basement = 5 × 8 = 40 ft

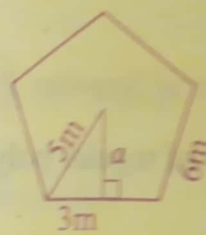
Area of basement = $\frac{1}{2}(40)(7) = 140 \text{ ft}^2$

Key Fact

Apothem is the length of perpendicular drawn from centre of a regular polygon to any side. It is also the radius of inscribed circle of a regular polygon.

EXERCISE 9.6

1. A lawn is in the shape of equilateral triangle. Find the perimeter of lawn if length of one side is 5m. Also find the cost of boundary wall of lawn @ Rs. 220 per metre.
2. Cricket ground in a village is in the shape of parallelogram. One side of ground is 65m long and distance between parallel sides having length 65m is 42m. Find the cost of planting grass @ Rs.10 per square metre.
3. Base of a minaret of a Masjid is built in the shape of regular pentagon as shown in the figure. Find the perimeter and area of base of minaret.



4. A plot in the shopping area is in the shape of regular hexagon. One side of the plot is 10m long. Find:
   (i) the cost of fencing the plot @ Rs.160 per metre.
   (ii) area of the plot.
   (iii) the cost of filling the plot @ Rs. 500 per m^2.
5. A room is in the shape of square having perimeter of 48 feet. Find:
   (i) the cost of carpeting the floor @ Rs. 350 per m^2.
   (ii) the inner area of each wall if room is 10 feet high.
   (iii) the cost of painting inner sides of the room @ Rs.100 per m^2.
6. A tile is in the shape of regular hexagon. Each side of the tile is one foot long. Find the perimeter of four tiles joined together as shown in the figure.

Locus

A locus is the set of points that satisfy a given condition. It is a path traced by a moving point which moves according to some given geometrical condition.

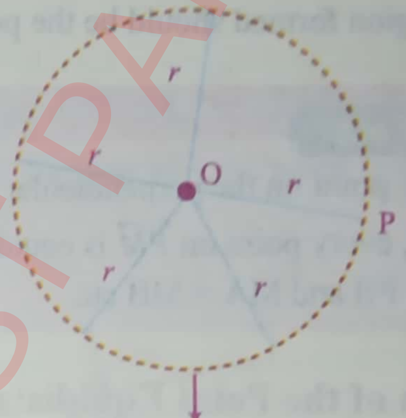
Key Fact

- Every point which satisfies the given geometrical conditions will lie on the locus.
- Every point lying on the locus will satisfy the given geometrical condition.
- The plural of 'locus' is 'loci'. The word locus is derived from the word location.

Locus of a Point from a Fixed Point

A circle with centre 0 and radius r cm is the locus of a point P moving in a plane in such a way that its distance from a fixed-point 0 is always equal to r cm.

This theorem helps to determine the region formed by all the points which are located at the same distance from a single point.

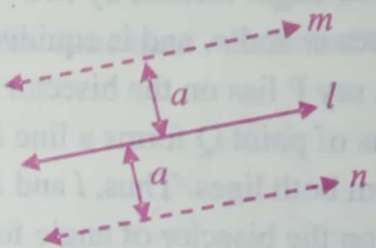


Key Fact

The locus of points in the plane equidistant from a given point is a ______, and the set of points in three-space equidistant from a given point is a sphere.

Locus of a Point from a Given Straight Line

Locus of a point that moves in such a way that its distance from a fixed line l is always equal to a cm. As clear from the figure, the locus of moving point is a pair of straight lines m and n each parallel to l and are located on either side of l at a distance a cm from the line l.

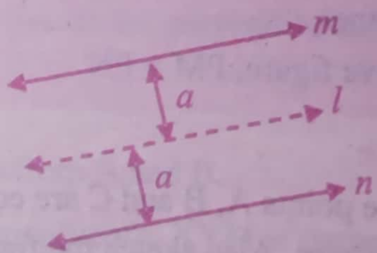


This theorem helps to find the region formed by all the points which are located at the same distance from a single line.

Key Fact

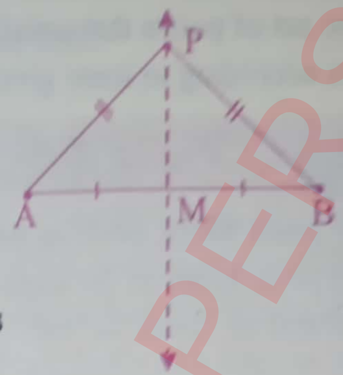
The locus of the point which is equidistant from the two parallel lines say m and n, is considered to be a line l parallel to both the lines m and n and it should be halfway between them.

This theorem helps to find the region formed by all the points which are at the same distance from the two parallel lines.



Locus of a Point Equidistant from Two Given Points

The locus which is equidistant from the two given points say A and B, is the perpendicular bisectors of the line segment that joins the two points.



Here, in the figure $\overleftrightarrow{PM}$ is the perpendicular bisector of $\overline{AB}$.

This theorem helps to determine the region formed by all the points which are located at the same distance from points A and B.

The region formed should be the perpendicular bisector of the line segment AB.

Key Fact

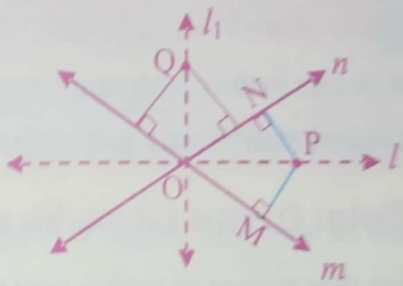
Every point on the perpendicular bisector of AB is equidistant from A and B.

Thus, every point on $\overleftrightarrow{PM}$ is equidistant from fixed points A and B.

PA = PB and MA = MB etc.

Locus of the Point Equidistant from Two Given Intersecting Lines

The locus of point which is equidistant from the two intersecting lines say m and n, is considered to be a pair of lines that bisects the angle formed by the two lines m and n.



In the figure, locus of point P forms a line l which bisects the angle formed by two intersecting lines m and n, and is equidistant from both lines. We say P lies on the bisector of angle formed by m and n.

Similarly, locus of point Q forms a line l_1 which bisects the angle formed by lines m and n, and is equidistant from both lines. Thus, l and l_1 are the pairs of lines that bisect the angle formed at O. We say Q lies on the bisector of angle formed by m and n.

This theorem helps to find the region formed by all the points which are located at the same distance from the two intersecting lines.

Key Fact

The locus of every point on the angle bisector of two intersecting lines, is equidistant from the lines.

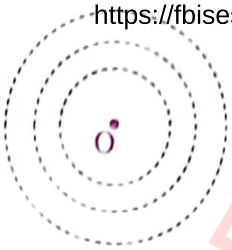
In the above figure, PM = PN

Example 20:

Loci of three points A, B and C are equidistant from a fixed point O. Prove that they form concentric circles. Also sketch the figure.

Solution:

Since the loci of the three points A, B and C are equidistant from a fixed point O, therefore, point O is the centre of circles formed by the movement of three points.



Hence the three circles formed are concentric circles.

Example 21:

A point P is moving parallel to a straight line l at a distance of 5m.

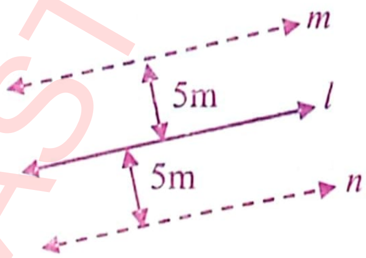
(i) Draw the locus of the path.

(ii) Prove that the point is moving in straight line.

(iii) How many paths are possible for the moving point P?

(iv) Another point Q is moving 8m away from the line l. Explain why the point Q is not the part of the locus?

Solution:



(i) Locus of the path of point P is possible along the lines m and n that are 5m apart from l.

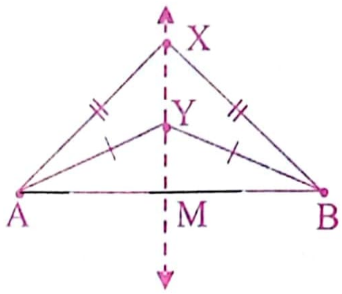
(ii) As the point P is moving parallel to straight line l, therefore it follows straight path too.

(iii) Two paths which are along m and n.

(iv) As the point Q is moving along straight line 8m away from l, therefore it is not part of locus of point P which is 5m away from l.

Example 22:

Figure shows two isosceles triangles XAB and YAB on the same base AB. Show the produced line XY bisects AB and is perpendicular to AB.



Solution:

Draw line through X and Y intersecting AB at M.

As ΔXAB is isosceles, therefore:

$XA = XB$ (i)

Which shows that X is equidistant from A and B and hence lies on the perpendicular bisector of AB. Similarly, ΔYAB is isosceles, therefore:

$YA = YB$ (ii)

Which shows that Y is equidistant from A and B and hence lies on the perpendicular bisector of AB.

This shows that XY is perpendicular bisector of AB.

Also, (i) and (ii) show that every point on XY is equidistant from A and B.

Since M lies on XY, therefore M is mid-point of AB and consequently XYM is perpendicular bisector of AB.

Example 23:

Take three non-collinear points and find a point O which is equidistant from these non-collinear points. Draw the locus of point passing through these three non-collinear points. What could be the name of this locus? Give the point O a specific name.

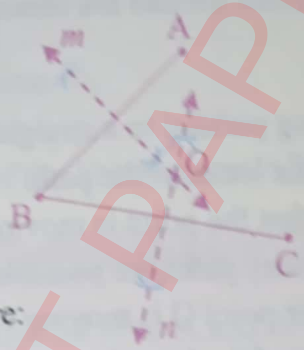
Solution:

Let A, B and C be three non-collinear points in the plane.

Join AB and BC.

Draw perpendicular bisector m of AB and n of BC.

Both bisectors meet at O.



As O lies on perpendicular bisector of AB, therefore:

$OA = OB$ (i)

Again, as O lies on perpendicular bisector of BC, therefore:

$OB = OC$ (ii)

$\Rightarrow$ $OA = OB = OC$ (from i and ii)

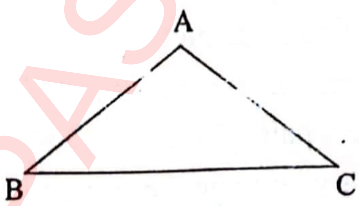
Hence O is equidistant from points A, B and C, and if we draw a circle with centre O, we get a locus of circle passing through three non-collinear points A, B and C called circumcircle.

Point O is called circumcenter.

EXERCISE 9.7

1. Draw AB = 6cm. Bisect AB at O and draw a locus of point P equidistant from O and above AB. What could be the name of locus?
2. Draw coordinate axes. Take two points A and B on x-axis and y-axis respectively at a distance of 4.5cm each. Draw a locus of points from A to B equidistant from origin. What is specific name of this locus? How many such loci can be drawn around the origin?
3. Draw a horizontal line l.
   - (i) Take a point T above l at a distance of 3cm and draw a locus of points through T parallel to l.
   - (ii) Now take a point Q below l at a distance of 3.2cm and draw a locus of points through Q parallel to l.
   - (iii) What is distance between both loci?
4. Diagram shows a circle with centre P. X and Y are two points on the circumference of the circle.
   - (i) Draw locus of points which are equidistant from X and Y through P.
   - (ii) Take another point Z on the locus outside the circle and draw another circle of radius PZ.
   - (iii) What is the relation of this circle with given circle?

5. Draw a line AB. Let P and Q be two points not on AB but coplanar with AB. Draw the locus of points from P to Q.
   (i) What is the name of that locus?
   (ii) Is there any point on the locus PQ if extended which lies on line AB?
   (iii) How can we take points P and Q such that no point of the above locus lies in line AB?
   (iv) How can we take points P and Q such that every point of line AB may lie on locus?
6. Draw an equilateral triangle PQR of suitable measurement.
   (i) Draw right bisectors of any two sides and locate a point A where both bisectors meet.
   (ii) Draw angle bisectors of any two vertices and locate a point B where both bisectors meet.
   (iii) What is the relation between locus of A and B?
7. Figure shows an isosceles triangle ABC. Prove that the locus of bisector of angle A is right bisector of side BC.



8. Draw three non-collinear points in the plane. Find the locus of the points which are equidistant form these three points. How many such points exist?
9. Take two lines AB and CD inclined at 60° intersecting at O.
   (i) Draw a locus of points which are equidistant from both lines.
   (ii) Draw bisector of 60°.
   (iii) What is relation between locus of points equidistant from lines and angle bisector?
   (iv) Draw bisector of adjacent angle at O find the relation between both angle bisectors.

KEY POINTS

- A mathematical statement is a meaningful composition of words that can be either true or false.
- A proof is a series of conjectures and axioms (postulates) and proved theorems that combine together to give a true result.
- An axiom is a mathematical statement which is assumed to be true without any proof.
- A statement that is believed to be true but its truth has not been proved is called a conjecture.

- A mathematical statement which can be proved through logical reasons is called a theorem.
- Two or more figures that have the same shape but not the same size are called similar figures.
- The ratio of areas of two similar figures is equal to the square of ratios of any two corresponding lengths of the figures.
- The ratio of volumes of two similar solids is equal to the cube of the ratio of any two corresponding lengths of the two solids.
- If all the sides and interior angles of the polygons are equal, they are known as regular polygons.
- If in a polygon at least one angle or one side is not equal in size then the polygon is called irregular.
- The sum of measures of all the interior angles of a polygon having n sides is:

$$(n-2)\times 180^\circ$$

- Sum of measures of exterior angles of a polygon is 360°.
- The locus of points in the plane equidistant from a given point is a circle.
- The locus which is equidistant from the two given points say A and B, is the perpendicular bisectors of the line segment that joins the two points.
- The locus of point which is equidistant from the two intersecting lines say m and n, is considered to be a pair of lines that bisects the angle formed by the two lines m and n.
- The locus of every point on the angle bisector of two intersecting lines, is equidistant from the lines.

MISCELLANEOUS EXERCISE 9

1. Encircle the correct option in the following.

(i) Which of the following is polygon?
(a) circle (b) pyramid (c) quadrilateral (d) sphere

(ii) Which of the following is regular polygon?
(a) kite (b) rhombus (c) rectangle (d) square

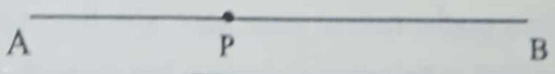
(iii) If two triangles are similar, their corresponding sides are.
(a) proportional (b) equal (c) congruent (d) parallel

(iv) What is the sum of interior angles for an irregular hexagon?
(a) 120° (b) 720° (c) 135° (d) 360°

(v) What is the sum of interior angles for a regular 12 sided polygon?
(a) 1800° (b) 2160° (c) 1980° (d) 360°

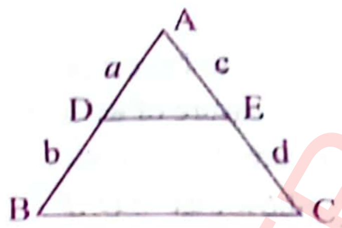
(vi) How many sides a regular polygon has if its exterior angle is 15?
(a) 20 (b) 21 (c) 24 (d) 25

(vii) In the figure, $AB = 21\ cm$ and



P divides AB in the ratio 3 : 4. What is length of $\overline{AP}$?
(a) $8\ cm$ (b) $10\ cm$ (c) $12\ cm$ (d) $9\ cm$

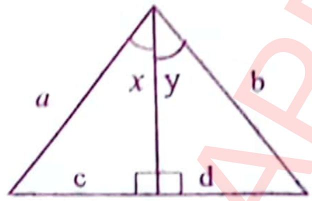
(viii) In the figure, $\frac{a}{b} = \frac{c}{d}$. Which one is true?



(a) $DE = BC$ (b) $DE > BC$

(c) $\overline{DE} \cong \overline{BC}$ (d) $\overline{DE} \parallel \overline{BC}$

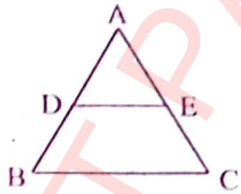
(ix) In the figure if $x = y$. Then the value of b is:



(a) $\frac{ac}{d}$ (b) $\frac{ad}{c}$

(c) $\frac{cd}{a}$ (d) $\frac{c}{cd}$

(x) In the figure, ΔABC is equilateral and $\overline{DE} \parallel \overline{BC}$, then ΔADE is

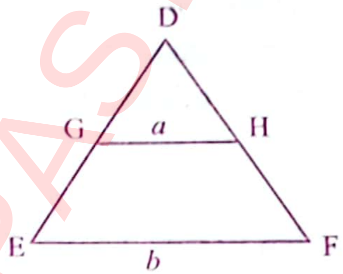


(a) right angled (b) scalene

(c) isosceles (d) equilateral

(xi) In the figure, $\overline{GH} \parallel \overline{EF}$.

Then $a : b = ?$



(a) $DG : DE$ (b) $DG : DH$

(c) $DG : GE$ (d) $DE : EF$

(xii) What is the sum of all the exterior angles of a 13-sided polygon whose one interior angle is equal to x°?

(a) $90^{\circ} + x$ (b) 360° (c) $360^{\circ} + x$ (d) $180^{\circ} + x$

(xiii) Which polygon has both its interior and exterior angles the same?

(a) pentagon (b) triangle (c) square (d) hexagon

(xiv) The formation or expression of an opinion or theory without sufficient evidence for proof is known as:

(a) axiom (b) conjecture (c) corollary (d) theorem

(xv) A mathematical statement that is proved true based on already accepted statements is called:

(a) axiom (b) conjecture (c) postulate (d) theorem

(xvi) A mathematical statement that is assumed to be true without proof is called:

(a) axiom (b) conjecture (c) postulate (d) theorem

(xvii) Two solids with equal ratios of corresponding linear measures:

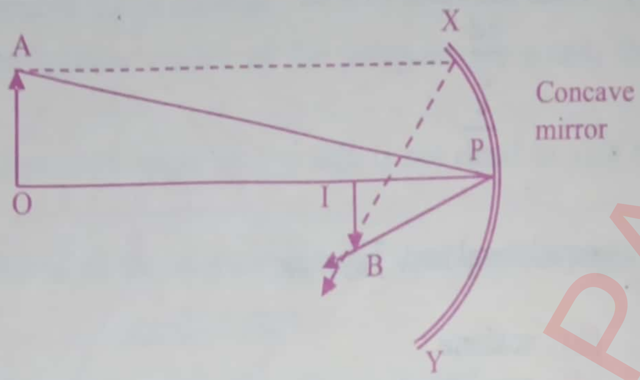
(a) are similar (b) are congruent

(c) have different area (d) have same volume

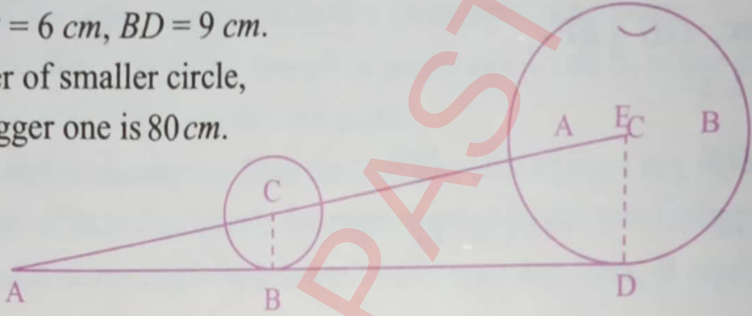
2. Find the exterior angle of a polygon with 6 sides.

3. Is it possible to have a polygon, in which sum of interior angles is 9 right angles?

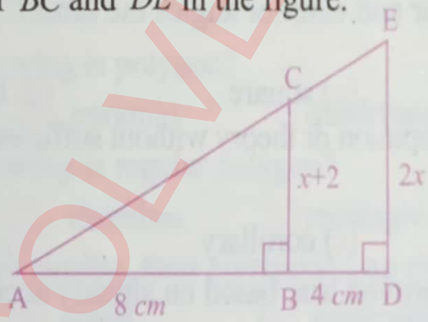
4. Is it possible to have a polygon whose sum of interior angles is $7200°$?
5. Find the measure of each angle of a regular nonagon.
6. In the figure XY is a concave mirror OA is object and IB is its image.
   (i) Show that $\Delta OAP \sim \Delta IBP$
   (ii) Find height of object if $IB = 2$ cm, $OP = 10$ cm, $IP = 4$ cm.



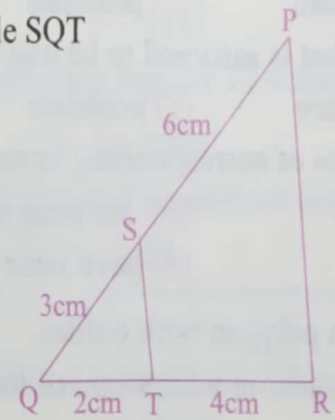
7. In the figure, $AB = 6$ cm, $BD = 9$ cm.
   Find the diameter of smaller circle,
   if diameter of bigger one is 80 cm.



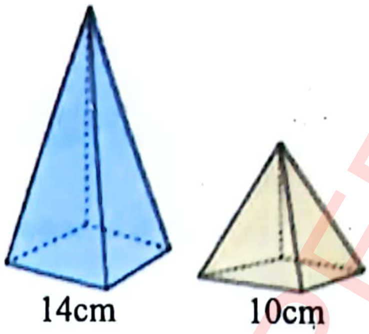
8. Prove that if vertex angles of two isosceles triangles are equal then the two **triangles are** similar.
9. Find the length of $\overline{BC}$ and $\overline{DE}$ in the figure.



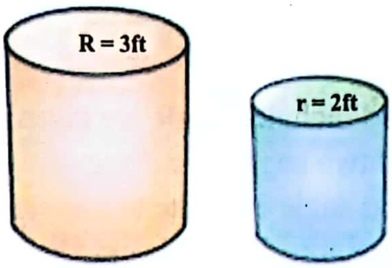
10. Triangles SQT and PQR are similar.
    Find the ratio of area of triangle SQT
    to that of triangle PQR.



11. The following pyramids are similar and larger pyramid has a surface area of $392\ cm^2$. What is the surface area of smaller pyramid?

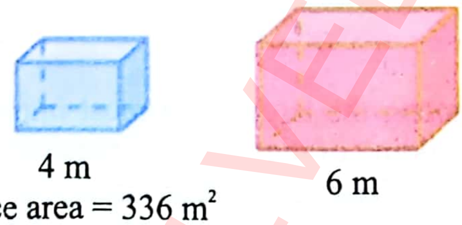


12. The two cylinders are similar. What is the volume of the larger cylinder if the volume of smaller cylinder is $40\ ft^3$.



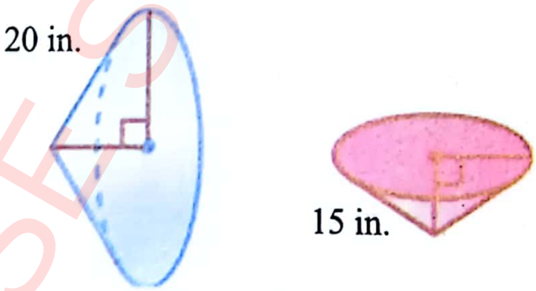
13. The following pairs of solids are similar. Find the surface area of red solid.

(i)



Surface area = $336\ m^2$

(ii)



Surface area = $1800\ in^2$.

UNIT
10

PRACTICAL GEOMETRY

In this unit the students will be able to:

- Construct a triangle when two sides and included angle are given.
- Construct a triangle when one side and two angles are given.
- Construct a triangle when two sides and angle opposite to one side are given.
- Draw angle bisectors, perpendicular bisectors, medians and altitudes of given triangle.
- Verify the concurrency of angle bisectors, perpendicular bisectors, medians and altitudes of given triangle.

Pyramids of ancient Egypt remain among the largest and most impressive structures constructed by any civilization. The building of the pyramids required a mastery of art, architecture, engineering and social organization at a level unknown before that time. Observe an image of those pyramids, shown in below and try to visualize their excellence in the field of construction.
Can you relate the geometrical shapes used in these pyramids with practical geometry?

Construction of Triangles

Construction of Triangle when Measures of Two Sides and Included Angle are Given

Example 1:

Construct a triangle ABC when $AB = 2.5\ cm$, $BC = 2.6\ cm$ and $\angle B = 75^o$.

Solution:

Steps of Construction:

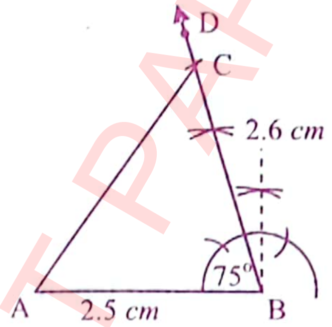
(i) Draw $AB = 2.5\ cm$.

(ii) Construct an angle of 75^o at point B with the help of ruler and compasses and draw $\overrightarrow{BD}$.

(iii) With center B, draw an arc of radius $2.6\ cm$ intersecting $\overrightarrow{BD}$ at C.

(iv) Join C to A.

ABC is required triangle.



Construction of a Triangle when Measures of One Side and Two Angles are Given

Example 2:

Construct a triangle PQR when $QR = 3.8\ cm$, $\angle P = 60^o$ and $\angle Q = 90^o$.

Solution: Angle R is required for construction. We know that:

$$\angle P + \angle Q + \angle R = 180^o$$
$$60^o + 90^o + \angle R = 180^o$$
$$\angle R = 30^o$$

Steps of Construction:

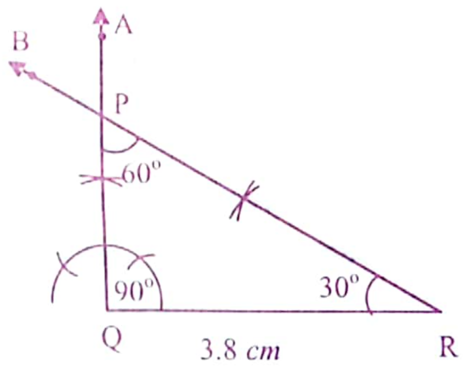
(i) Draw $QR = 3.8\ cm$.

(ii) Construct an angle of 90^o at Q and draw $\overrightarrow{QA}$.

(iii) Construct an angle of 30^o at R and draw $\overrightarrow{RB}$.

(iv) Both $\overrightarrow{QA}$ and $\overrightarrow{RB}$ intersect at point P.

Thus, PQR is required triangle.



Construction of a Triangle when Measures of Two Sides and Angle Opposite to One of the Sides are Given (Ambiguous Case)

In such construction, we are not sure that how many triangles can be constructed.

∴ The above case is also called ambiguous case.

Let us solve some examples to explain this case.

Example 3:

Construct a triangle ABC such that AB = 5.3 *cm* and BC = 4.8 *cm* and $\angle A = 60^\circ$.

Solution:

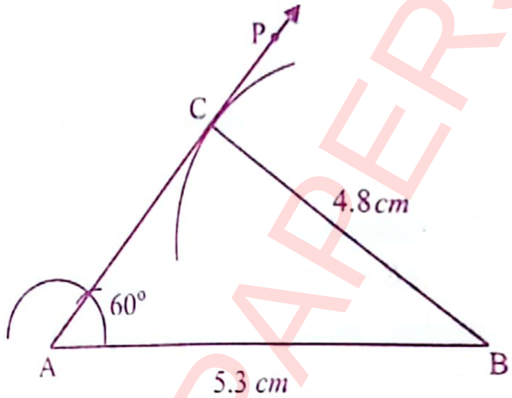
Steps of Construction:

(i) Draw AB = 5.3 *cm*.

(ii) Construct $\angle BAP = 60^\circ$.

(iii) With centre *B*, draw an arc of radius 4.8 *cm* which intersects $\overrightarrow{AP}$ at point C.

ABC is required triangle.



Example 4:

Construct a triangle having lengths of two sides 4.8 *cm* and 4.3 *cm* and angle of 60° opposite to the side 4.3 *cm* long.

Solution:

Steps of Construction:

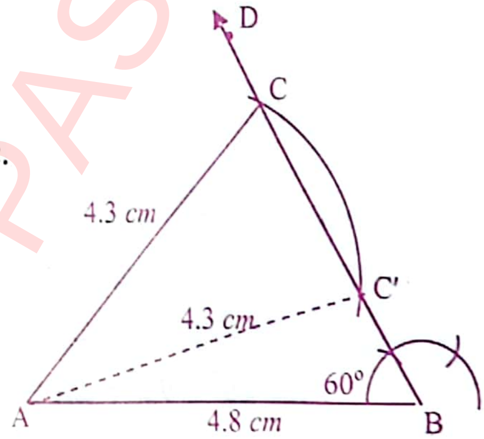
(i) Draw a line segment say *AB* of length 4.8 *cm*.

(ii) Construct $\angle ABD = 60^\circ$.

(iii) With center *A*, draw an arc of radius 4.3 *cm* intersecting $\overrightarrow{BD}$ at points *C* and *C'*.

(iv) Join *C* and *C'* to *A*.

Thus, two triangles *ABC* and *ABC'* are constructed.



Example 5:

Construct a triangle having lengths of two sides as 2.3 *cm* and 5.4 *cm* and angle of 30° is opposite to the side 2.3 *cm*.

Solution:

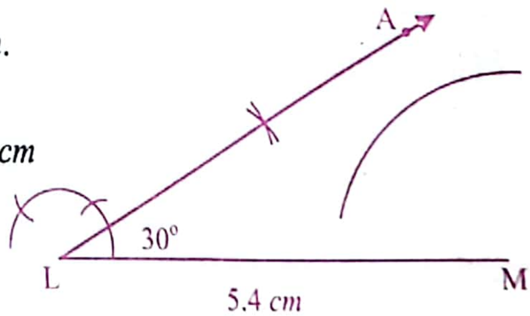
Steps of Construction:

(i) Draw a line segment *LM* of length 5.4 *cm*.

(ii) Construct $\angle MLA = 30^\circ$.

(iii) With centre *M*, draw an arc of radius 2.3 *cm* which does not intersect $\overrightarrow{LA}$.

$\therefore$ Construction of triangle is not possible.



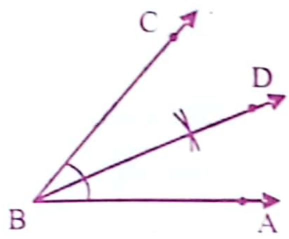
EXERCISE 10.1

1. Construct the following triangles.
   (a) ΔABC when $AB = 5.8\ cm$, $AC = 4.2\ cm$, $\angle A = 90^\circ$
   (b) ΔLMN when $LM = 4\ cm$, $MN = 4.7\ cm$, $\angle M = 120^\circ$
   (c) ΔPQR when $PQ = 7.2\ cm$, $\angle P = 45^\circ$, $\angle Q = 75^\circ$
   (d) ΔXYZ when $XY = 5\ cm$, $\angle X = 30^\circ$, $\angle Z = 105^\circ$
2. Construct the following triangles where possible.
   (a) $AB = 6.8\ cm$, $BC = 8.1 cm$, $\angle A = 90^\circ$
   (b) $DE = 5.2\ cm$, $DF = 4\ cm$, $\angle E = 45^\circ$
   (c) $QR = 7.0\ cm$, $PQ = 5.6\ cm$, $\angle R = 75^\circ$
   (d) $XY = 3.8\ cm$, $XZ = 5\ cm$, $\angle Y = 60^\circ$
   (e) If two sides of lengths 5.7 *cm* and 7.5 *cm* are given and angle of 105° is opposite to the side of length 7.5 *cm*.
   (f) If two sides of length 6.1*cm* and 3.8 *cm* are given and angle of 30° is opposite to the side of length 3.8 *cm*.

Bisectors of Angles of a Triangle

A ray which divides a given angle into two equal parts is called bisector of that angle. In the figure $\overrightarrow{BD}$ divides angle *ABC* into two equal angles *ABD* and *CBD*.

So $\overrightarrow{BD}$ is the bisector of angle *ABC*.



If we draw the bisectors of all the three angles of a triangle, they meet at a single point inside the triangle. We say that *the angle bisectors of a triangle are concurrent.*

Example 6: Construct a triangle *ABC* such that $AB = 4.7\ cm$, $BC = 4.5\ cm$ and $AC = 4.6\ cm$. Draw bisectors of three angles. Are they concurrent?

Solution: Steps of Construction:

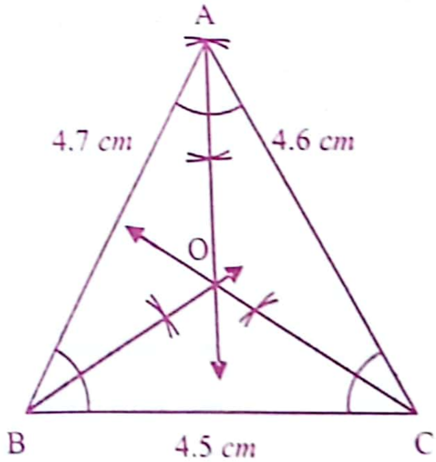
(i) Draw $BC = 4.5$ cm.

(ii) With centre *B*, draw an arc of radius 4.7 *cm*.

(iii) With centre *C*, draw another arc of radius 4.6 *cm* intersecting first arc at *A*. Join *A* to *B* and *C*. *ABC* is the required triangle.

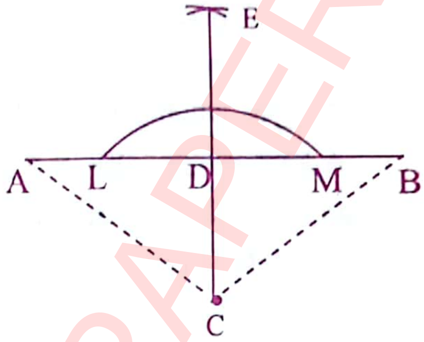
(iv) Draw bisectors of $\angle A$, $\angle B$ and $\angle C$ which meet at point *O*.

$\therefore$ *Angle bisectors of triangle are concurrent.*



Altitudes of a Triangle

A perpendicular line segment from a vertex to opposite sides of triangle is called altitude of triangle.



In the adjoining figure, $\overline{CE} \perp \overline{AB}$. Also, CD is the distance between point C and $\overline{AB}$.
If we join C to A and B, a triangle ABC is formed.
We say that CD is altitude of triangle ABC with respect to base AB.
Similarly, we can draw altitudes with respect to other sides. All the three altitudes meet at a single point.
$\therefore$ *The altitudes of a triangle are concurrent.*

Example 7:

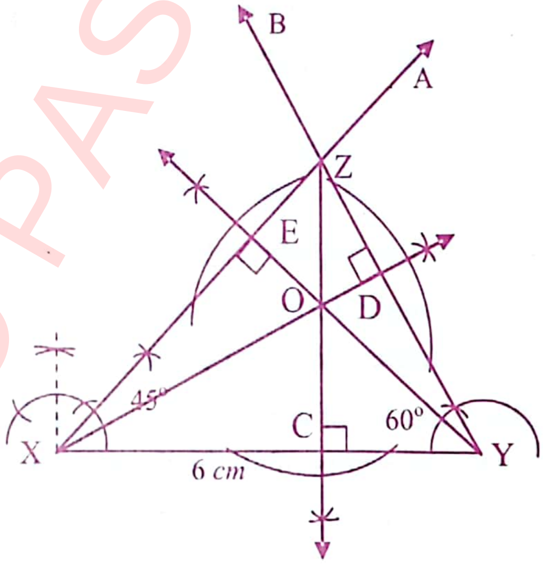
Construct ΔXYZ when:

$$XY = 6\ cm,\ \angle X = 45^\circ,\ \angle Y = 60^\circ$$

Draw altitudes of triangle. Are they concurrent?

Solution:

Steps of Construction:



(i) Draw $XY = 6\ cm$.

(ii) Construct $\angle YXA = 45^\circ$ and $\angle XYB = 60^\circ$.

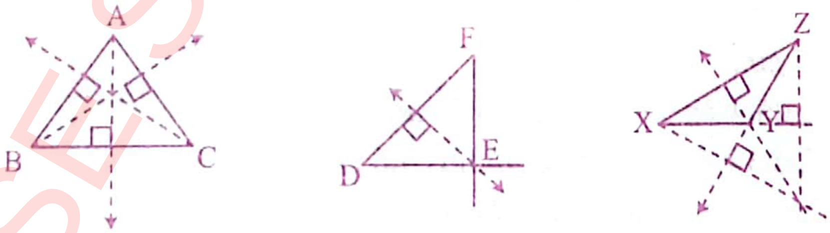
(iii) $\overrightarrow{XA}$ and $\overrightarrow{YB}$ meet at Z.

(iv) Draw altitudes $\overline{XD} \perp \overline{YZ}$, $\overline{YE} \perp \overline{XZ}$ and $\overline{ZC} \perp \overline{XY}$.

(v) The three altitudes meet at O.
$\therefore$ Altitudes of a triangle are concurrent.

Key Fact

Look at the figures.

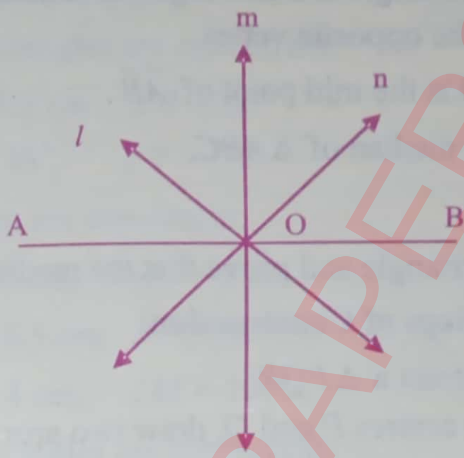


- Altitudes of an acute angled triangle meet inside the triangle.
- Altitudes of a right-angled triangle meet at the vertex of right angle.
- Altitudes of an obtuse angled triangle meet outside the triangle.

Perpendicular Bisectors of Sides of Triangle

Bisector is a line which divides a line segment into two equal parts. In the figure, lines l, m and n are the bisectors of $\overline{AB}$. Among the bisectors, the line m is perpendicular to $\overline{AB}$. So, m is a perpendicular bisector or right bisector of $\overline{AB}$. Point O is mid point of $\overline{AB}$.

i.e. OA = OB



Example 8:

Construct a ΔLMN when $LM = 4.1\ cm,\ MN = 4\ cm$ and $\angle M = 90^o$. Draw perpendicular bisectors of sides of triangle. Are they concurrent?

Solution:

Steps of Construction:

(i) Draw $LM = 4.1\ cm$.

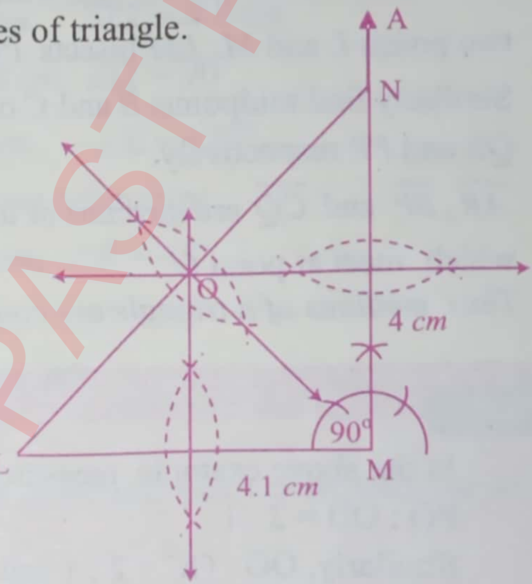
(ii) Construct $\angle LMA = 90^o$.

(iii) With centre M, draw an arc of radius 4 cm intersecting $\overrightarrow{MA}$ at N.

(iv) Join N to L.

(v) Draw right bisectors of $\overline{LM}$, $\overline{MN}$ and $\overline{LN}$ which meet at point O.

$\therefore$ *Right bisectors of sides of a triangle arc concurrent.*

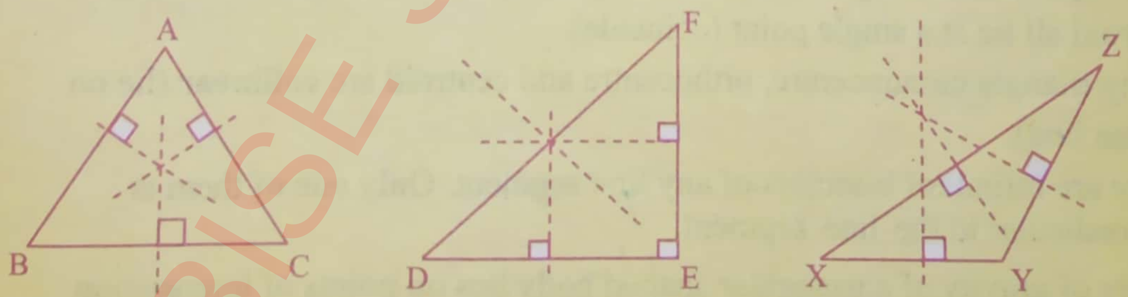


Note: Look at the figures below.

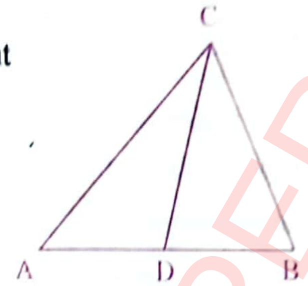
(a) Right bisectors of an acute angled triangle meet inside the triangle.

(b) Right bisectors of a right-angled triangle meet at the mid point of the hypotenuse.

(c) Right bisectors of an obtuse angled triangle meet outside the triangle.



Medians of a Triangle



Median of a triangle is a line segment joining the mid point of a side to the opposite vertex.

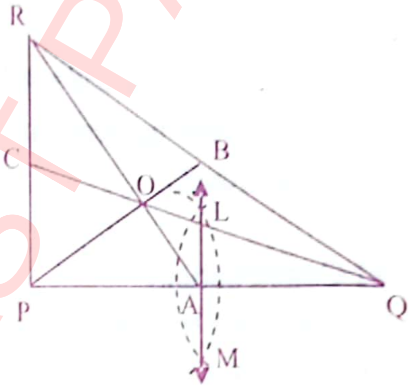
In ΔABC, D is the mid point of $\overline{AB}$.

$\therefore$ $\overline{CD}$ is the median of ΔABC.

Example 9:

Construct a triangle and prove that the medians of a triangle are concurrent.

Solution: Steps of Construction:



(i) Construct a ΔPQR

(ii) With centres P and Q, draw two arcs of equal radius (more than half of PQ), which meet at two points L and M. $\overleftrightarrow{LM}$ bisects $\overline{PQ}$ at A.

(iii) Similarly find midpoints B and C of the sides QR and PR respectively.

(iv) $\overline{AR}$, $\overline{BP}$ and $\overline{CQ}$ are medians of triangle which meet at point O.

Thus, medians of a triangle are concurrent.

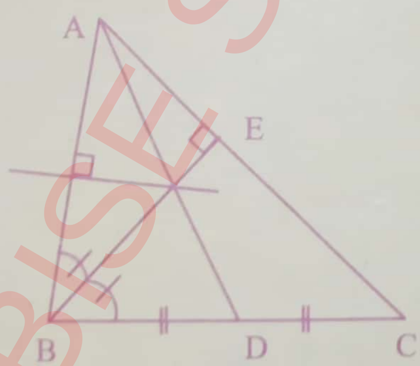
Key Fact

- In the above example, measure PO and OB. You will see that PO : OB = 2 : 1.
  Similarly, QO : OC = 2 : 1 and RO : OA = 2 : 1
  ∴ *Medians of a triangle trisect each other.*
- The point where angle bisectors of triangle meet is called **inscribed** centre or **in-centre**.
- The point where altitudes of a triangle meet is called **orthocentre.**
- The point where perpendicular bisectors of a triangle meet is called **circumcentre**.
- The point where medians of a triangle meet is called **centroid**.
- In an equilateral triangle inscribed centre, circumcentre, orthocentre and centroid all lie at a single point (coincide).
- In any triangle circumcentre, orthocentre and centroid are **collinear** (lie on a same line).
- There are infiniteof bisectors of any line segment. Only one of them is perpendicular to the line segment.
- Centre of gravity of a triangular shaped body lies on points of intersection of medians.

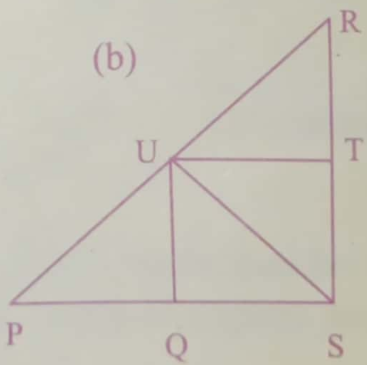
EXERCISE 10.2

1. Show that angle bisectors of the following triangles are concurrent.
   (a) ΔABC when $AB = 5.8\ cm$, $BC = 4.9\ cm$, $\angle B = 60^\circ$
   (b) ΔXYZ when $XY = 5.5\ cm$, $\angle X = 45^\circ$, $\angle Y = 75^\circ$
2. Show that altitudes of the following triangles are concurrent.
   (a) ΔABC when $AB = 4\ cm$, $BC = 5\ cm$, $AC = 6\ cm$.
   (b) ΔPQR when $PQ = 4.6\ cm$, $QR = 6.5\ cm$, $\angle P = 90^\circ$
   (c) ΔLMN when $LM = 4.2\ cm$, $MN = 4\ cm$, $\angle M = 105^\circ$
3. Show that right bisectors of the following triangles are concurrent.
   (a) ΔXYZ when $XY = 4.5\ cm$, $YZ = 5\ cm$, $ZX = 4.8\ cm$
   (b) ΔPQR when $PQ = 4\ cm$, $QR = 5.8\ cm$, $\angle Q = 90^\circ$
   (c) ΔDEF when $DE = 5\ cm$, $EF = 4\ cm$, $\angle E = 120^\circ$
4. Show that medians of the following triangles are concurrent.
   (a) ΔABC when $AB = 5.8\ cm$, $BC = 5\ cm$, $\angle B = 45^\circ$
   (b) ΔDEF when $DE = 6\ cm$, $\angle D = 90^\circ$, $\angle E = 30^\circ$
5. Construct a right triangle ABC such that $\angle B = 90^\circ$
   (a) Find its orthocentre, where does it lie?
   (b) Find circumcentre M of the triangle. Is M the mid-point of hypotenuse AC. Is $BM = CM$?
6. Construct an obtuse angled triangle DEF. Find its (a) circumcentre
   (b) orthocentre. Check whether they lie inside or outside the triangle?
7. Construct an isosceles triangle and find its
   (a) centroid
   (b) inscribed centre.
8. Recognize altitudes, perpendicular bisectors and medians in the following triangles.

(a)



(b)



KEY POINTS

- If two sides and their non included angle are given, more than one triangle can be constructed but sometimes construction of such triangles becomes impossible.
- Only one triangle can be constructed if:
  (a) three sides are given.
  (b) two sides and their included angle are given.
  (c) two angles and any side are given.
- The case in which we are not sure about the number of triangles to be constructed is called an *ambiguous* case.
- Angle bisectors of a triangle are always concurrent inside the triangle.
- A line segment which joins vertex of a triangle to its opposite side perpendicularly is called its altitude.
- Altitudes of acute angled, right angled and obtuse angled triangles are concurrent inside, at the vertex of right angle and outside the triangle respectively.
- Perpendicular bisectors of acute angled, right angled and obtuse angled triangle are concurrent inside, at the mid point of hypotenuse and outside the triangle respectively.
- Medians of triangles are concurrent inside the triangle.
- The point of concurrency of
  - Angle bisectors of a triangle is called in-centre (inscribed centre)
  - Altitudes of a triangle is called orthocenter.
  - Perpendicular bisectors of a triangle is called circum-centre.
  - Medians of a triangle is called centroid.
  - Medians of a triangle divides each median in the ratio 2:1.

MISCELLANEOUS EXERCISE 10

1. Encircle the correct.

(i) A line which bisects an angle into two equal parts is

(a) right bisector (b) angle bisector (c) altitude (d) median

(ii) Which of the following is divided by a point called midpoint?

(a) an angle (b) a line (c) a line segment (d) a ray

(iii) A line passing through mid point and perpendicular to a side of a triangle is called

(a) right bisector (b) angle bisector (c) mid point (d) altitude

(iv) Two sides making arms of a right angle in right triangle are its

(a) altitudes (b) medians (c) vertices (d) bisectors

(v) Right bisectors of a right triangle meet at mid point of

(a) medians (b) altitude (c) side (d) hypotenuse

(vi) Medians of a triangle are

(a) collinear (b) concurrent (c) perpendicular (d) parallel

(vii) In an equilateral triangle angle bisectors, medians, altitudes and right bisectors

(a) are parallel (b) are perpendicular (c) coincide (d) are collinear

(viii) Right bisectors of equiangular triangle are its

(a) angle bisectors (b) altitudes (c) medians (d) All a, b, c

(ix) If three lines meet at a point, they are called

(a) intersecting (b) perpendicular (c) collinear (d) concurrent

(x) Medians of a triangle intersect each other in the ratio.

(a) 1 : 2 (b) 1 : 3 (c) 3 : 1 (d) 2 : 3

2. Construct the following triangles.

(i) ΔPQR when $PQ = 6\ cm$, $\angle P = 30^\circ$, $\angle R = 90^\circ$.

(ii) ΔXYZ when $XY = 5.6\ cm$, $XZ = 5.2\ cm$, $\angle Y = 60^\circ$.

(iii) ΔABC when $BC = 7cm$, $AC = 4.3\ cm$, $\angle B = 45^\circ$.

3. Construct a right isosceles triangle whose hypotenuse is 6 *cm*.

4. Construct an equilateral triangle *ABC*. Find its

(a) incentre (b) circumcentre (c) orthocentre (d) centriod

Do they coincide?

5. A student wants to find incentre of an equilateral triangle but he does not know how to bisect the angles. Can he find incentre by using any other method? Explain your answer by taking an example.

6. Construct a triangle and find its centre of gravity. (Hint: Find centroid)

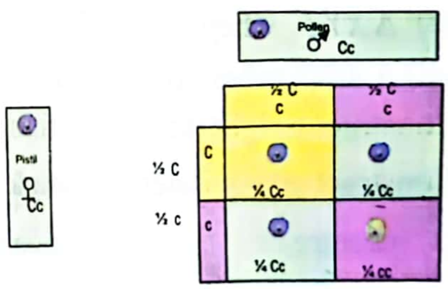
UNIT
11

BASIC STATISTICS

In this unit the students will be able to:

- Construct grouped frequency table, Histogram(unequal intervals) and frequency polygon.
- Calculate Arithmetic mean, Median, Mode of grouped data.
- Calculate weighted mean.
- Related real life word problems
- Probability of single events.
- Probability of complementary events.
- Relative frequency as an estimate of probability.
- Expected frequency.
- Related real life word problems.

The famous Biologis Gregor Mendel demonstrated that pea plant characteristic were transmitted as discrete units from parent to off springs i.e in a genetic cross, the probability of a pea plant of purple flowr with a pea plant of yellow flowrs, the dominate trait is expressed in 3 out of 4 of the offspring and recessive trait is expressed in one out of 4 as shown in the adjoining figure.



Statistic plays vital role in our day to day life as in our lives we have to come across frequent decisions based on some relevant past experiences or the related statistics already available. For instance we may use statistics in eficient budgeting of our family , in finding the best prices for sale or purchase of products of our interest, to predict risks in our business, to estimate best time to perform various routine activities. Online shopping is an easily understandable example of this era where the customers see reviews of others about a product and make decision to purchase or not to purchase. If more than 50% reviews are in the favour it may be predicted that the product will be out of stock soon. Similarly , to estimate the chances of failure or success of a project, probability has great application. Quantifying and estimating investment and risk assessment in a project are very important for financial planning , investment comparison and decision making.

THINK MORE

- Hadiya is on the holy trip of umrah while her stay she observed the best time to perform TWWAF E KAABAH PAK when the crowd is least is soon after Fajar prayer when of the people go back to rest. What do you guess for another such time

11.1 Frequency Distribution

In statistics it is very important to represent a data in a manageable way so that it becomes easy to understand, analyze and to draw useful results. This useful presentation of data can be made by various methods. Grouped frequency distribution is one of the most important methods to represent the quantitative data. Data can be classified as:

Raw Data / Ungrouped Data

Data which has not been arranged in a systematic order is called raw data or ungrouped data.

Grouped Data

A tabular arrangement of data in which various items / values are arranged or distributed into some classes and the number of items falling in each class is stated is called grouped data or a frequency distribution.

Let us try to understand the above two categories of data with the help of an example. The points scored by 50 participants of Maths Olympiad out of 100 are recorded below.

10 18 38 15 30 27 41 39 16 50 50 45 34 37 34 09 12 45 19 23 29 31 28 17 08
19 25 39 42 48 11 30 21 34 37 50 46 08 15 15 16 21 28 32 37 41 19 29 48 50

This presentation of data is called ungrouped data. It is difficult to draw any useful conclusions from the above data. To analyze the above results easily , it is better to distribute the students in some groups according to their marks as follows.

Marks obtained	No. of students
1 – 10	4
11 – 20	12
21 - 30	11
31 – 40	11
41 – 50	12
	Total = 50

From the table it is easy to get the following results.
There are 4 participants whose scores are from 1 to 10.
There are 12 participants whose scores are from 11 to 20.
There are 11 participants whose scores are from 21 to 30.
There are 11 participants whose scores are from 31 to 40.
There are 12 participants whose scores are from 41 to 50.
34 participants scored more than 20 points.
Data represented in this form is called grouped data.
To convert an ungrouped data into grouped form, the students should be familiar with the following terms.

(i) Number of Groups

A number of groups depends upon the size of data. There is no hard and fast rule for a number of groups, but it is usually between 5 and 15.

(ii) Class Limits

A group or class is defined by two numbers. The smaller number is called lower class limit and larger one is called upper class limit. In above example first group is (1 – 10), 1 is its lower class limit and 10 the upper class limit. Both the lower and upper limits are included in the respective groups.

(iii) Class Boundaries

In above example the values / marks between 10 and 11 are not included in any group. If marks of a student are 10.5, then above mentioned classes are not suitable. To include such values we should write the classes as

0.5 – 10.5, 10.5 – 20.5, 20.5 – 30.5, 30.5 – 40.5, 40.5 – 50.5.

The participants having scores between 0.5 and 10.5 including 0.5 are considered in the first group. The participants having scores 10.5 will be in second group. The largest value of 0.5 – 10.5 and smallest value of 10.5 – 20.5 are same.

0.5 and 10.5 are lower and upper boundaries of first group. The class boundaries can be formed from class limits as follows. First of all find difference between the lower class limit of second class and upper class limit of first class. Then subtract half of this difference from lower limits of all classes and add it to the upper limits of all classes.

Adjustment factor = $\frac{1}{2}$(lower limit of a class – upper limit of the preceeding class) = 0.5

To understand the difference between class limits and class boundaries, look at the following table.

scores (class intervals)	Lower limits	Upper limits	Lower boundaries	Upper boundaries	Class boundaries
1 – 10	1	10	1 – 0.5 = 0.5	10 + 0.5 = 10.5	**0.5 – 10.5**
11 – 20	11	20	11 – 0.5 =10.5	20 + 0.5 = 20.5	**10.5 – 20.5**
21 – 30	21	30	21 – 0.5 = 20.5	30 + 0.5 = 30.5	**20.5 – 30.5**
31 – 40	31	40	31 – 0.5 = 30.5	40 + 0.5 = 40.5	**30.5 – 40.5**
41 – 50	**41**	**50**	**41 – 0.5 = 40.5**	**50 + 0.5 = 50.5**	**40.5 – 50.5**

(iv) Size of Class Interval

Size of a class interval is the difference of lower limits or upper limits of two consecutive classes. It is usually denoted by 'h'.
e.g. in 1 – 10, 11 – 20, 21 – 30, the size of class intervals is 10.

(v) Class Marks / Mid Values

It is the average of lower and upper limits or boundaries of a class. Class marks are denoted by x.
Class mark of 1 – 10 is $\frac{1+10}{2} = 5.5$ and that of 11 – 20 is $\frac{11+20}{2} = 15.5$

Note:
1. The difference of class marks of two consecutive classes is equal to the size of class interval h.
2. Class boundaries may be derived from the class marks x and size of class interval h by using the formula $(x \pm \frac{h}{2})$.

(vi) Frequency

Number of repetitions of an individual value in an ungrouped data is called its frequency. In a grouped data, number of values present in a group is called frequency of group. In Table 1 frequency of group 1 – 10 is 4, because there are four students whose obtained marks are between 1 and 10 both inclusive.

(vii) Cumulative Frequency

The sum of frequency of a class and the frequencies of all the preceding classes is called cumulative frequency.

10.1.1 Construction of Frequency Distribution Table

We can construct a frequency distribution table in two ways.
(i) Tally bar method (ii) Direct observation method

Tally Bar Method

Step I: Decide how many groups should be made in the table.

Step II: Find approximate size of class interval by the formula $h \approx \frac{x_l - x_s}{n}$,

Where h = size of class interval
x_l = largest value in ungrouped data
x_s = smallest value in ungrouped data
n = approximate number of classes

Step III: Draw a table and write the class limits in first column according to the size of class interval.

Step IV: Choose the values from ungrouped data one by one and draw small tally marks in second column against the respective group. If 5 or more values fall in a same group, then draw 5^{th} tally mark diagonally. It will make the counting of tally marks easier.

Step V: Count tally marks of each group and write their number against the respective class in third column (column of frequencies).

Step VI: Find sum of all frequencies and check that it must be equal to the number of values present in ungrouped data i.e
number of total items in ungrouped data (n) = sum of frequencies in grouped data ($\sum f$)

Example 1:

The following are masses in pounds of 40 boys at a gym.
147, 138, 164, 150, 132, 144, 125, 149, 157, 146, 158, 140, 136, 148, 152, 144, 168, 126, 138, 176, 163, 126, 154, 165, 146, 173, 142, 147, 135, 153, 140, 135, 161, 145, 135, 142, 150, 156, 145, 128.
Make a frequency distribution table using an appropriate size of class interval by tally bar method.

Solution:

Step I: Let us distribute the data in 5 classes.

Step II: $x_l = 176, x_s = 125, n \approx 5.$

$$h = \frac{x_l - x_s}{n} = \frac{176-125}{5} = 10.2 \approx 11$$

Step III:

Masses in pounds	Tally marks	Frequencies
125 – 135	𝍸 |||	8
136 – 146	𝍸 𝍸 |||	13
147 – 157	𝍸 𝍸 |	11
158 – 168	𝍸 |	6
169 – 179	||	2
Total		**40**

> **History a Mystery**
>
> Tally sticks are the ancient tools used for recording numerical data. These elongated pieces of bone, wood or ivory were marked with counts or lines. Marco Polo have documented the use of tally sticks in regions like Yunnan (China).

Direct Observation Method

In this method the values of ungrouped data are written in second column of table against the respective group instead of tally marks. The remaining process of construction of frequency table by direct method is same as that of tally bar method. When ungrouped data are not arranged in any order, then it is better to use tally bar method.
In example 1, the frequency table by direct method can be constructed as follows.

Masses in pounds	Observation	Frequencies
125 – 135	132, 125, 126, 126, 135, 135 , 135, 128	**8**
136 – 146	138, 144, 146, 140, 136, 144, 138, 146, 142, 140, 145, 142, 145	**13**
147 – 157	147, 150, 149, 157, 148, 152, 154, 147, 153, 150, 156	**11**
158 – 168	164, 158, 168, 163, 165, 161	**6**
169 – 179	176, 173	**2**
Total	—	**40**

Discrete Frequency Distribution

If data is not in fractions and some values occur most often to constitute the whole data, then data is called a discrete data there is no need of class boundaries and size of class interval is 1.

Example 2:

A cubical die is rolled 30 times and numbers of dots on front face recorded each time are as follows.

2, 6, 2, 3, 2, 4, 5, 4, 3, 1, 2, 6, 5, 3, 4, 6, 2, 2, 4, 6, 5, 1, 3, 4, 6, 5, 1, 1, 6, 3

Make a discrete frequency distribution. Also make a column of cumulative frequency

Solution:

No. of dots	Tally marks	Frequencies	Cumulative frequency
1	||||	4	4
2	~~||||~~ |	6	4+6=10
3	~~||||~~	5	4+6+5 =15
4	~~||||~~	5	4+6 +5+5= 20
5	||||	4	4+6 +5+5+4=24
6	~~||||~~ |	6	4+6 +5+5+4+6=30
Total	—	**30**	

11.1.2 Construction of Histogram

Different types of charts and graphs are used to represent a data effectively. It becomes very easy to get a quick idea about the data and to analyze by means of graphs and charts. In histogram, data are represented by adjacent rectangles having bases along horizontal axis (marked off by class boundaries) and areas proportional to the class frequencies. If the class interval sizes is the same, the heights of rectangles are proportional to the class frequencies as you have done earlier in grade 8. If the class interval size is unequal, then the heights of rectangles are to be adjusted.

Construction of Histogram with Unequal Size of Class Intervals

If sizes of class intervals are unequal, then to make areas of rectangles of a histogram proportional to their frequencies, the heights of rectangles have to be adjusted. In a histogram area of rectangle is proportional to class frequency.

$$\text{Area} \propto \text{frequency} \quad \text{or} \quad \text{height} \times \text{width} \propto \text{frequency}$$

$$\Rightarrow \text{height} \propto \frac{\text{frequency}}{\text{width}}$$

Thus height of rectangle may be found by dividing the frequency by width of class interval. The ratio $\frac{\text{frequency}}{\text{width}}$ is called **frequency density**.

To make a histogram with unequal sizes of class intervals, we plot class boundaries on x – axis (horizontal) and frequency densities on y – axis (vertical).

Example 3: The heights in cm of 44 students of a school is given below.

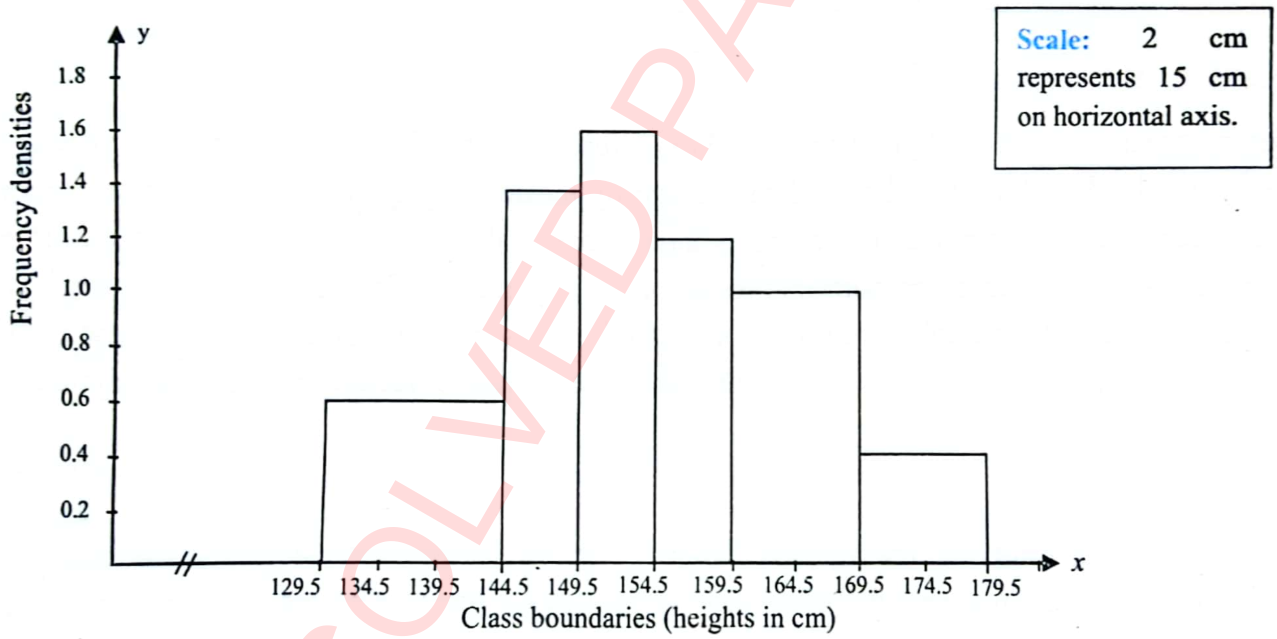
Height	130–144	145–149	150–154	155–159	160–169	170–179
No. of students	9	7	8	6	10	4

Represent the data by a histogram.

Solution:

Height (cm)	f	Size of class interval	F.D	Class boundaries
130 – 144	9	15	$9 \div 15 = 0.6$	129.5 – 144.5
145 – 149	7	5	$7 \div 5 = 1.4$	144.5 – 149.5
150 – 154	8	5	$8 \div 5 = 1.6$	149.5 – 154.5
155 – 159	6	5	$6 \div 5 = 1.2$	154.5 – 159.5
160 – 169	10	10	$10 \div 10 = 1$	159.5 – 169.5
170 – 179	4	10	$4 \div 10 = 0.4$	169.5 – 179.5
Total	**44**	—	—	—

Title: A histogram showing heights (cm) of 44 students in a secondary school.



11.1.3 Construction of a Frequency Polygon

Polygon is a many sided closed figure. To draw a frequency polygon mid points (class marks) are marked on the horizontal axis and frequencies are on the vertical axis. Points are plotted with class frequencies and their corresponding class marks. These points are joined by straight line segments. To complete a closed polygon we add extra classes at both ends with zero frequencies. In this way two extra points on x-axis are obtained on both ends. The points are joined by their nearer plotted points. Finally a closed frequency polygon is obtained. It is easy to draw a frequency polygon on a histogram. To do so we find mid points of the top edges of rectangles and join them by segments. Then find two extra points on both ends on x-axis to complete the closed polygon.

Example 4:

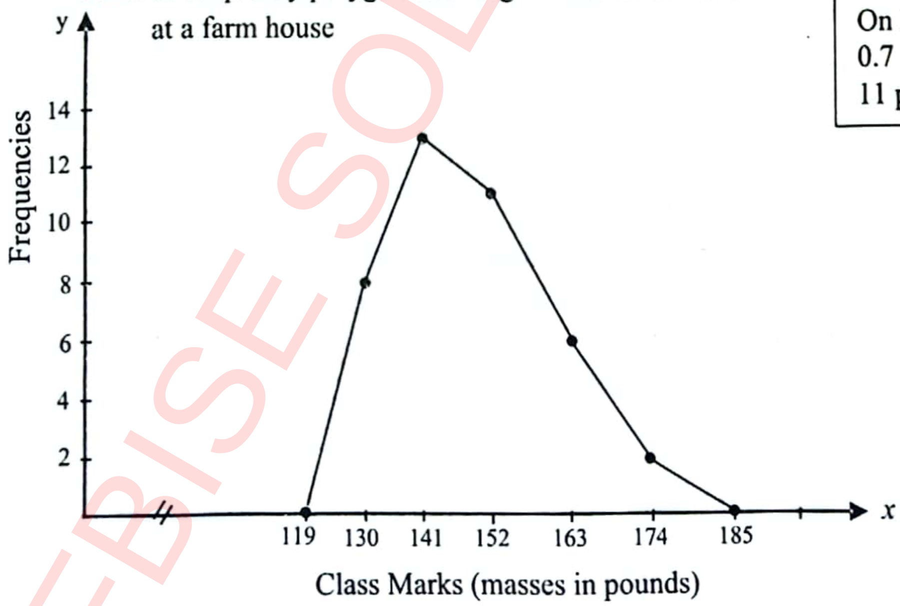
At a farm house there are 40 lambs , their masses are recorded as given below. Represent the following frequency distribution by a frequency polygon.

Masses in pounds	No. of lambs
125 – 135	8
136 – 146	13
147 – 157	11
158 – 168	6
169 – 179	2
Total	**40**

Solution:

Masses	Frequency	Class Marks
114 – 124	0	**119**
125 – 135	8	**130**
136 – 146	13	**141**
147– 157	11	**152**
158 – 168	6	**163**
169 – 179	2	**174**
180 – 190	0	**185**
Total	40	

Title: A frequency polygon showing masses of 40 lambs at a farm house



Scale:
On horizontal axis 0.7 cm represents 11 pounds

Example 5:

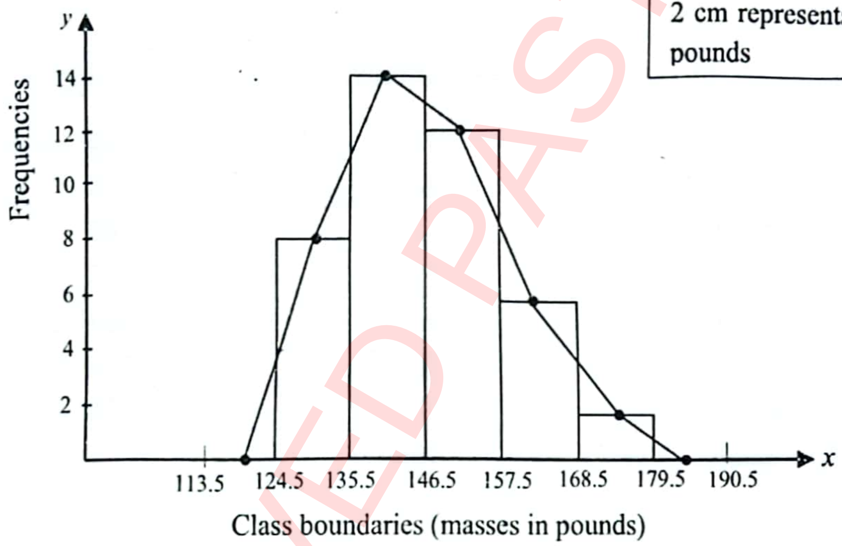
Use the frequency distribution in example 4 and draw a frequency polygon on a histogram.

Solution:

Masses	Frequency	Class boundaries	Class Marks
125 – 135	8	124.5 – 135.5	130
136 – 146	14	135.5 – 146.5	141
147 – 157	12	146.5 – 157.5	152
158 – 168	6	157.5 – 168.5	163
169 – 179	2	168.5 – 179.5	174

Title: A frequency polygon on a histogram showing masses of 40 lambs in pounds

Scale: On horizontal axis 2 cm represents 11 pounds



Check Point

Read the following passage carefully and count number of letters in each word.
"Statistics is a mathematical science pertaining to the collection, analysis, interpretation or explanation and presentation of data. It is applicable to a wide variety of academic disciplines from the physical and social sciences to the humanities".
Make a discrete frequency distribution for
(i) the number of letters in each word, (ii) the number of vowels in each word.

EXERCISE 11.1

1. The given table shows number of regular readers of a library who have completely read the books.

Class Interval (books)	Frequency (No. of readers)
1 – 10	5
11 – 20	4
21 – 30	8
31 – 40	9
41 – 50	2
51 – 60	2

(i) What is the total number of readers in the data?
(ii) Which group contains highest number of readers?
(iii) Which group contains least number of readers?
(iv) What is the lower limit of the last class?
(v) What is the lower boundary of the last class?
(vi) What is the size of the class interval?
(vii) Find class marks (mid points) of all groups.

2. Number of gratitude cards made by 80 student of class 9 is given below.
79, 60, 74, 59, 55, 98, 61, 67, 89, 71, 71, 46, 63, 66, 69, 42, 75, 62, 71, 77, 78, 65, 87, 57, 78, 91, 82, 73, 65, 94, 48, 87, 62, 81, 63, 66, 65, 49, 45, 51, 69, 56, 84, 93, 63, 60, 68, 51, 73, 54, 50, 88,76, 93, 48, 70, 40, 76, 95, 57, 63, 94, 82, 54, 89, 64, 77, 94, 72, 69, 51, 56,67,88,81,70, 81, 54, 66, 87

(i) Prepare a frequency table to represent the above data using the classes 40 – 49 etc.
(ii) Add cumulative frequency column in the table .
(iii) How many students made less than 70 cards?
(iv) What percent of students made less than 50 cards?
(v) Which group has the greatest frequency?
(vi) What is the size of the class interval?

3. The number of medals won by 45 players in a certain sports festival is given as :
0, 2, 1, 0, 1, 2, 3, 5, 6, 3, 2, 1, 3, 4, 2, 6, 1, 5, 2, 4, 3, 0, 1, 2, 3, 0, 0, 2, 3, 4, 1, 5, 6, 2, 4, 5, 1, 3, 4,6, 2, 3, 1, 2, 5 .
Prepare a discrete frequency distribution. Also make a column of cumulative frequencies.

4. The ages of workers in a factory were recorded as below.

Ages in Years	20 – 24	25 – 29	30 – 34	35 – 39	40 – 44	45 – 49
No. of workers	5	16	12	10	8	4

Draw a frequency polygon to represent the data.

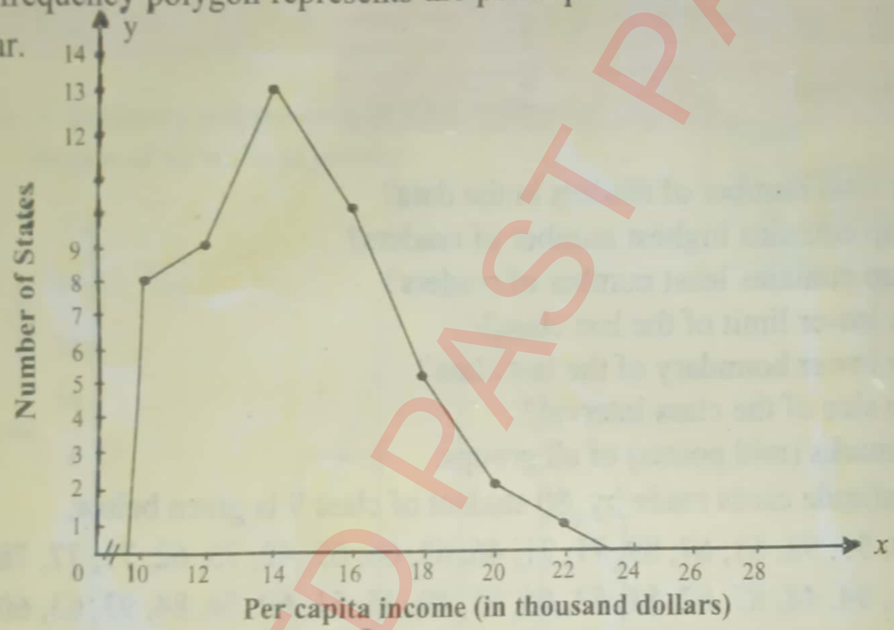
5. Represent the following data by a histogram.

Class Interval	0 – 5	5 – 15	15 – 30	30 – 40	40 – 45
Frequency	5	12	16	10	2

6. In a saving group, there are 400 members, and the number of savings certificates held by them is shown in the following table. Construct a histogram of the distribution of saving certificates.

Class Interval	1 – 50	51–100	101–150	151–200	201–300	301–400	401–500
No. of members	10	15	30	40	120	100	85

7. The following frequency polygon represents the per-capita income of some states, recorded in a particular year.



(i) What is the total number of states?
(ii) What is the number of states which have the least per-capita income?
(iii) What is the number of states having highest per-capita income?
(iv) What percent of states have per-capita income \$14000 and \$16000?
(v) How many states have per-capita income \$18000 and above?
(vi) Represent the frequency polygon by a frequency table.

8. The masses measured to the nearest kg of 50 boys are given in the frequency distribution as :

Mass (kg)	60 – 64	65 – 69	70 – 79	80 – 89	90 – 94	95 – 99
f	2	6	12	14	10	6

(i) Construct a histogram. (ii) Construct a frequency polygon.

9. At a sale, number of items placed according to the price range is given below:

Price(Rs)	600 – 999	1000 – 1999	2000–2499	2500 – 2999	3000– 3499
f	50	70	75	65	70

Construct a histogram

11.2 Measures of Central Tendency

In this unit we are looking for a better representation of data. The purpose of summarizing a data is to draw meaningful results, to analyze the data and to compare two or more data having same characteristics. When the data have been arranged into a frequency distribution the information contained in it is easily understood. The data becomes clearer when it is represented by graphs. Still we need a simpler way to represent a data in more efficient way to analyze and understand better. A single value that represents a set of data is called an average. Since the averages tend to lie in the center of a data, they are called measures of central tendency. They are also called measures of location because they tend to be located at the center of the data. Here we will discuss 3 very common measures of central tendency.

(i) Arithmetic Mean
(ii) Median
(iii) Mode

11.2.1 (a) Arithmetic Mean

There are two sections A and B of class 9 in a public school consisting of 10 and 12 students respectively. The marks secured by each student in a test out of 100 are given below :

A : 35, 40, 90, 45, 50, 80, 70, 65, 55, 50
B : 25, 50, 30, 35, 60, 45, 45, 85, 90, 65, 45, 35

It is required to compare the performance of two sections and we have to decide that which one of them is better than the other. It is difficult and lengthy to compare the individual performance of students. Let us see the total marks secured by the students in each section. The total of both sections are same. It is not good to say that performance of both sections is same, because they have different number of students. We have to compare sum of scores and numbers of students in each section simultaneously. To do this we will find the ratio of sum of scores and number of students in each section separately and then we will compare the two results obtained.

In section A, sum of scores: number of students = 610:10 = 61
In section B, sum of scores: number of students = 610:12 = 50.83
As 61 > 50.83 so average performance of section A is better than that of section B.
"The ratio of sum of values of a data and number of values is called an arithmetic mean."

Arithmetic Mean for Ungroup Data

The values of data are usually denoted by x. In case of more than one data, their values are denoted by x, y or z etc. The sum of the values of the data 'x' is denoted by "$\sum x$" and number of values is denoted by n. The arithmetic mean of data x is symbolized a $\bar{x}$ and by above definition:

$$\bar{x} = \frac{\Sigma x}{n} = \frac{\text{sum of values}}{\text{numbers of values}}$$

Arithmetic Mean for Group Data

If values of data occur more than once, the sum of values can be found by using **frequencies e.g.**
If $x = 2, 3, 1, 2, 2, 3, 2, 3, 4, 2, 5, 2, 3, 4, 5, 2, 1, 3, 5, 1$
Then, $\sum x = 2+3+1+2+2+3+2+3+4+2+5+2+3+4+5+2+1+3+5+1 = 55$
In this data frequency of 1 is 3 i.e. 1 occurs three times. Similarly frequencies of 2, 3, 4, 5, are 7, 5, 2, 3 respectively.

$$\sum x = 1\times 3 + 2\times 7 + 3\times 5 + 4\times 2 + 5\times 3 = 3 + 14 + 15 + 8 + 15 = 55$$

Also n = number of values = sum of frequencies = 3 + 7 + 5 + 2 + 3 = 20

So, $\bar{x} = \dfrac{\sum x}{n} = \dfrac{55}{20} = 2.75$

The above data can be summarized in frequency distribution as follows:

x	1	2	3	4	5	**Total**
f	3	7	5	2	3	$\sum f = 20$
fx	$1\times 3 = 3$	$2\times 7=14$	$3\times 5=15$	$4\times 2=8$	$5\times 3=15$	$\sum fx = 55$

The arithmetic mean is $\bar{x} = \dfrac{\text{sum of values}}{\text{numbers of values}} = \dfrac{\sum fx}{\sum f} = \dfrac{55}{20} = 2.75$

The arithmetic mean for a grouped data can also be found by using class marks 'x' **and class** frequencies 'f' with the help of formula $\bar{x} = \dfrac{\sum fx}{\sum f}$, if size of class interval is greater than 1

Example 6: Find the arithmetic mean of the following data

Marks	0 – 10	10 – 20	20 – 30	30 – 40	40 – 50
No. of students	2	5	7	3	8

Solution:

Marks	f	Class marks	fx
0 – 10	2	5	10
10 – 20	5	15	75
20 – 30	7	25	175
30 – 40	3	35	105
40 – 50	8	45	360
Total	$\sum f = 25$	–	$\sum fx = 725$

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{725}{25} = 29$$

11.2.1 (b) Median

When data is arranged in some order, it can be represented by a single value that lies exactly at the centre of data. "The central or middle most value of an **arranged data** is called its median represented by $\tilde{x}$ read as x tilde .

Median for Ungroup Data

- If the number of values of a data is odd, then the middle value of the arranged data is the median.
- If the number of values of a data is even, then there are two middle values, the arithmetic mean of these two middle values is called median of the data. These definitions of median can be expressed for an ungrouped data as following formulas.

(i) If n (number of values) is odd, then median $= \frac{n+1}{2}$ th value of the data

(ii) If n is even, then Median = Arithmetic mean of the two middle values

$$= \frac{1}{2}\left\{n\text{th value} + \frac{n+2}{2}\text{th value}\right\}$$

Example 7:

The selling prices of the seven articles sold by a shopkeeper in rupees were 1075, 1050, 2050, 1030, 2045, 1090 and 1530. Find the median selling price of these articles.

Solution:

Arranging these values in the ascending order as 1030, 1050, 1075, 1090, 1530, 2045, 2050.

Here $n = 7$(odd), so

$$\text{Median} = \frac{n+1}{2}\text{th value} = \frac{7+1}{2}\text{th value} = \frac{8}{2}\text{th value} = 4^{th}\text{ value} = 1090$$

So, median selling price of seven articles is Rs.1090

Example 8:

Find the median of the following set of values.

40, 35, 45, 60, 50, 35, 40, 45, 35, 50

Solution: Arrange the numbers in ascending order as 35, 35, 35, 40, 40, 45, 45, 50, 50, 6

Here $n = 10$ (even), so

Median = $\tilde{x}$ = half of sum of two middle most values

$$= \frac{1}{2}\left\{\frac{n}{2}\text{th value} + \frac{n+2}{2}\text{th value}\right\}$$

$$= \frac{1}{2}\left\{\frac{10}{2}\text{th value} + \frac{10+2}{2}\text{th value}\right\}$$

$$= \frac{1}{2}(5\text{th value} + 6\text{th value})$$

$$= \frac{1}{2}(40+45) = \frac{85}{2} = 42.5$$

Median for Grouped Data

In a grouped data or frequency distribution the median is taken as $\frac{n}{2}th$ value. To find median for a grouped data we first find the group which contains the median. To locate the median group we form cumulative frequency table. The formula used to find median of a group data is

Median= $\tilde{x} = l + \frac{h}{f}(\frac{n}{2} - c)$

Where l = lower class boundary of median group

h = size of class interval

f = frequency of median group

$n = \Sigma f$ = Number of items

c = Cumulative frequency of class preceding the median class.

Example 9: Marks obtained by 100 students in an examination are given below,

Marks	10 – 19	20 – 29	30 – 39	40 – 49	50 – 59
No of students	5	25	40	20	10

Find the median marks.

Solution: Median = $\frac{n}{2}$ th value = $\frac{100}{2}$ th value = 50^{th} value.

Marks	f	Class Boundaries	c.f
10 – 19	5	9.5 – 19.5	5
20 – 29	25	19.5 – 29.5	c = 30
30 – 39	f = 40	29.5 – 39.5	70
40 – 49	20	39.5 – 49.5	90
50 – 59	10	49.5 – 59.5	100

→ **median group** (30 – 39)

From the above table, it is clear that 50^{th} value lies in the group 29.5 – 39.5. So 29.5 – 39.5 is the median group.

Median = $l + \frac{h}{f}(\frac{n}{2} - c)$ where $l = 29.5$, $h = 10$, $n = 100$, $f = 40$, c = 30

$$\text{Median} = 29.5 + \frac{10}{40}\left(\frac{100}{2} - 30\right)$$

$$= 29.5 + \frac{1}{4}(50 - 30)$$

$$= 29.5 + \frac{20}{4}$$

$$= 29.5 + 5 = 34.5 \text{ marks}$$

Danger zone !!!
Most of the students take c from the median group which gives the negative result . remember c is taken from the group which comes before median group

11.2.1 (c) Mode

The number of pairs of shoes sold in a day according to their sizes at a shoe store are given below.

Sizes	6	7	8	9	10	11
No. of pairs sold	14	25	45	50	30	16

According to the data, the most commonly used shoes are of size 9. In other words we can say that the average size of shoes sold in a day at a shoe store is 9. The average found in this way is called mode. "The most frequent (repeated) value of a data is called mode of the data." If two or more values occur the same numbers of times but more frequently than any of the other values, then there is more than one mode. In this respect, mode differs from mean and median because they have unique values. If each value of data occurs same numbers of times, then there is no mode.

Example 10: Find mode of following data sets.

(i) 30 , 25 , 40 , 30 , 45 , 25 , 30 , 40 , 30

(ii) 1 , 2 , 3 , 2 , 1 , 3 , 1 , 2

(iii) 125 , 130 , 115 , 125 , 135 , 120 , 135 , 130 , 120

Solution:

(i) As 30 is the most frequent value so it is mode of the data.The data is called unimodal.
(ii) As the values 1&2 occur equal number of times, so there are 2 modes. This data is called bimodal data.
(iii) As the values 120 , 125 and 130 occur two times each and frequencies of remaining values are lesser. So 120, 125 and 130 are threemodes of the data.This data is called trimodal data.

Mode for Group Data

To find mode for a frequency distribution or a group data, we first find the modal class. A class or group having largest number of items (frequency) is called modal group and it contains mode of the data. The mode of a grouped data can be found by the formula.

$$\text{Mode} = l + \frac{(f_m - f_l)}{2f_m - f_l - f_2} \times h$$

Where l = lower class boundary of the modal class.
f_m = frequency of the modal class (the highest frequency)
f_1 = frequency of the class preceding the modal class
f_2 = frequency of the class following the modal class
h = size of class interval

Note: If first group of the distribution have the largest frequency, then $f_1 = 0$ and if modal group is the last group of frequency distribution, then $f_2 = 0$. But these are very rare cases.

Example 11:

Fifty workers of a factory are distributed in 7 groups according to their wages in rupees per hour.

Wages in Rupees	25-29	30-34	35-39	40-44	45-49	50-54	55-59
No of workers	2	4	8	20	6	6	4

Find mode of the distribution.

Solution:

Wages	f	Class boundries
25 – 29	2	24.5 – 29.5
30 – 34	4	29.5 – 34.5
35 – 39	8	34.5 – 39.5
40 – 44	20	39.5 – 44.5 → modal class
45 – 49	6	44.5 – 49.5
50 – 54	6	49.5 – 54.5
55 – 59	4	54.5 – 59.5

As 20 is the largest frequency, so (39.5 – 44.5) is the modal group.

$$l = 39.5, f_m = 20, f_1 = 8, f_2 = 6, h = 5$$

$$\text{Mode} = l + \frac{(f_m - f_1)}{2f_m - f_1 - f_2} \times h$$

$$\text{Mode} = 39.5 + \frac{(20-8)}{2\times 20 - 8 - 6} \times 5$$

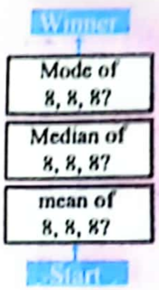
$$= 39.5 + \frac{12}{40-14} \times 5$$

$$= 39.5 + \frac{12}{26} \times 5 = 39.5 + \frac{60}{26}$$

$$= 39.5 + 2.31 = 41.81 \text{ Rupees}$$

Math Play Ground

1. Make such hopscotch in the play ground.
2. Ask a player to start hoping, doing calculations in each box.
3. Decide the winner if he completes without falling.



Example 12:

Find median and mode for the following discrete frequency distribution.

x	1	2	3	4	5
f	3	4	5	2	2

Solution:

Here $n = \Sigma f = 16$

Median = class containing $\frac{n}{2}$th observation= $\frac{16}{2}$th observation = 8th observation

x	f	c.f.
1	3	3
2	4	7
3	5	12 → 8^{th} observation
4	2	14
5	2	16

So Median = 3 (8^{th} observation lies in the group where c.f. is 12.)

Hence Mode = most frequent observation = 3

11.2.1 (d) Empirical Relation between Mean, Median and Mode

If any two of mean, median and mode of a data are given, then we can find an approximate value of the third measure of central tendency by using an empirical relation.

$$\text{mode} = 3\ \text{median} - 2\ \text{mean}$$

Or $$\text{median} = \frac{1}{3}(\text{mode} + 2\ \text{mean})$$

Or $$\text{mean} = \frac{1}{2}(3\ \text{median} - \text{mode})$$

11.2.2 Weighted Mean

If it is required to find the average of some values which are not equally important, then we assign them certain numerical values to express their relative importance. These numerical values are called weights. To find arithmetic mean of data whose values are not equally important, we use the weights instead of frequencies.

If x denotes the items of a data and w denotes their respective weights, then arithmetic mean is

$$\bar{x}_w = \frac{\sum wx}{\sum w}$$

Example 14: The following are the monthly expenditure of a house according to their weights.

Items	Expenditure (Rs)	Weight relative importance
Food	8,000	20
House rent	5,000	10
Clothing	1,000	6
Utility bills	900	4
Education	600	2
Miscellanies	100	1

Find the weighted mean.

Solution:

Items	x = Expenditure (Rs)	w = Weight	wx
Food	8,000	20	160000
House rent	5,000	10	50000
Clothing	1,000	6	6000
Utility bills	900	4	3600
Education	600	2	1200
Miscellanies	100	1	100
Total	—	**43**	**220900**

$$\text{Weighted mean, } \bar{x}_w = \frac{\sum wx}{\sum w} = \frac{220900}{43} = \text{Rs.}5137.21$$

EXERCISE 11.2

1. Find arithmetic mean for the following data by definition
   (i) $x = 2, 4, 6, 8, 10, 12$
   (ii) $y = 3, 4, -1, 7, -8, -5, 0$
   (iii) $z = 0, 4, 8, 12, 16, 20, 24, 28$
   (iv) $u = 3.1, 4.2, 5.3, 6.4, 7.5, 8.6, 9.7, 10.8$
   (v) $v = 5, 5, 5, 5, 5, 5, 5, 5$
2. Following are the scores made by two batsmen A and B in a series of innings:

A	12	15	6	73	7	19	199	36	84	29
B	47	12	76	48	4	51	37	48	13	0

Find arithmetic mean of scores of both players and state who is better as run getter?

3. Marks of four students of class 9 in four subjects out of 100 are given below. The weights of subjects according to their importance are also given.

Subjects	Weights	Marks			
		Hanzala	Musa	Zaki	Saleh
Mathematics	4	70	80	90	95
Physics	3	95	60	55	70
Chemistry	2	80	65	50	60
Biology	1	55	45	65	60

Find weighted mean of marks of each student.

4. Following frequency distribution represents number of liters of fuel consumed against the distance in km. Find arithmetic mean of this data set.

Class intervals	0 – 5	5 – 10	10 – 15	15 – 20	20 – 25
Frequencies	5	7	10	6	2

5. Find arithmetic mean from the following data of number of a detergent packs sold against its mass in kg .

Classes(kg)	1 – 9	10 – 18	19 – 27	28 – 36	37 – 45
No. of packs	10	15	18	15	12

6. Find median for the following data.
   (i) 1, 4, 2, 5, 3, 7, 6
   (ii) ± 2, ± 4, ± 6, ± 8
   (iii) 0, 1, – 2, – 3, 4, 5, 6, 3
   (iv) 4, 3, 1, – 3, 2, – 3, 3, 4, 1
   (v) 4, 4, 4, 4, 4, 4

7. In a practical of geometry and shapes students are required to construct a circle of any radius between 0cm and 2cm. the data given below is recorded.

Radoius (xcm)	1.8	1.6	1.4	1.2	1.0	0.8	0.6	0.4	0.2
No. of circles (f)	6	9	15	24	33	20	13	7	5

Find median radius of the circle.

8. At an old age house, there are people of different age groups as given below.

Classes intervals	40 – 50	50 – 60	60 – 70	70 – 80	80 – 90
f	3	8	11	5	4

Find an average median age of people residing in that old age house.

9. Find mode for the following data.

(i) 1, 2, 3, 4, 5, 6

(ii) 2, 4, 2, 3, 2, 5, 3, 2, 5, 4, 2

(iii) 120, 130, 140, 225, 125, 225, 120

(iv) 2, 4, 3, 5, 5, 3, 4, 2

10. Find mean, median and mode for the following discrete frequency distribution.

x	0	1	2	3	4	5
f	3	5	9	4	2	1

11. Find modal value for the following frequency distributions.

(a)

Classes	1 – 4	5 – 8	9 – 12	13 – 16	17 – 20
f	9	8	10	6	7

(b)

Classes	10 – 15	15 – 20	20 – 25	25 – 30	30 – 35
Frequency	7	6	3	2	1

11.3 Probability

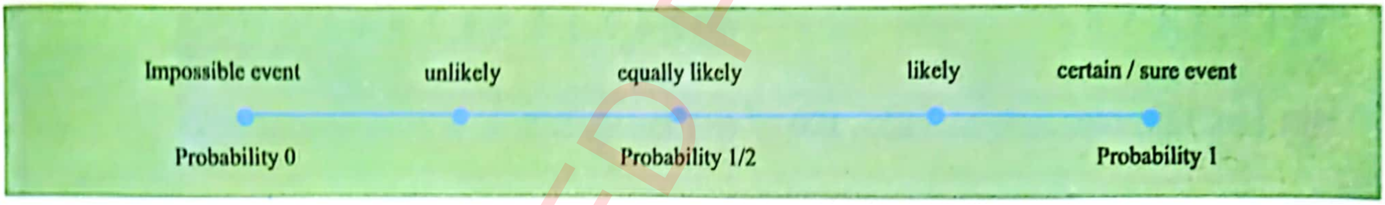
We often come across such situations where we have to make guess about occurrence or non occurrence of something that it would happen or not. For instance we often make such statements :

- 40% students of class 9 are predicted to be absent tomorrow because of severe heat wave.
- You will probably win Maths Olympiad this year .
- I hope you will get at least 90% marks in your board exam .

In these statements one can notice the uncertainty or doubt of occurrence. These are few of the guesses made in our daily life for the possibility of an event that the event is likely to happen or not. Probability is synonymous to possibility. Simply speaking the measure of chance of an event is called probability and its vlues are from 0 to 1.For example

- Probability of a new prophet after Hazrat Muhammad (SAW) is 0.
- Probability that Haleema will win the toss among Haleema, Hajra &Haani is 1/ 3.
- Probability of sun setting in west is 1.

In these statements we have seen that probability is between 0 and 1 both inclusive. This range can also be presented pictorially for better understanding as



In previous classes you have learnt about various terms necessary to understand the problems of probabilitty i.e.
Experiment, trial, outcome , sample space, event, impossible event, sure event, equally likely ,likely, unlikely, mutually exclusive events, exhaustive events, independent and dependent events etc. Here we will discuss single event and events not happening in a bit detail .

11.3.1 Single Event Probability

An event which has single sample point is called a single event or simple event or singular event. Eg in tossing a coin the sample space is {H, T}, where E_1 = {H} and E_2 = {T}, as only one out of 2 outcomes is possible . Here both E_1 and E_2 are single events.

Example 15: Write sample space for single roll of a die and enlist all the events. Discuss whether any compound event is present in this case.

Solution: Here we notice if a single die is rolled once then any one of the outcome may occur from 1, 2, 3, 4, 5 and 6. Therefore S = {1, 2, 3, 4, 5, 6}
Now all possible events are {1}, {2}, {3}, {4}, {5 }, {6}
from events listed above, its evident that all events are single .So no compound event in this case.

The probability of such events is obtained by the formula:

$$P(E) = \frac{No.of\ favourable\ outcomes}{total\ no.of\ outcomes}$$

Example 16: Write sample space for single spin of a spinner of 3 primary colours and enlist all the events. Discuss probabilities of all events .

Solution: Here S = {blue, red , green}, the pointer of the spinner can stop at only one of the colour so E_1= {blue}, E_2 = {red}, E_3 = {green}

$$P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$$

Example 17: Ali picked a book at random from a library shelf which had 3 history books of grade 6, 4geography books grade7, 9 science books of grade 8, 3 science books of grade 7, 6 geography books of grade 8 and 5 science books of grade 6. Find the probability of getting:

a. a science book b. a geography book c. a book of grade 7

d. a book of grade 9 e. a book of mathematics

Solution: Here total number of books is 30.

a. total science books are 17 .·So $P(E_1)=\frac{17}{30}$

b. total geography books are 10. So $P(E_2) = \frac{10}{30} = \frac{1}{3}$

c. total books of grade 7 are 7. So $P(E_3) = \frac{7}{30}$

d. there is no book of grade 9. So $P(E_4) = \frac{0}{30} = 0$

e. there is no book of mathematics. So $P(E_5) = \frac{0}{30} = 0$

11.3.2 Probability of an Event not occurring

An event which describes the non occurrence of another event is called non occurring event.

If 4 boys Affan , Jaffar, Hassan and Maaz are playing ONO, then probability of winning of each boy is $\frac{1}{4}$. If the event of winning of jaffar is represented by J, then $P(J) = \frac{1}{4}$. Now the event that Jaffar won't win is represented by J′ and $P(J') = 1-P(J) = 1-\frac{1}{4} = \frac{3}{4}$

The two events in which one shows non occurrence of other are called complementary events.

REMEMBER: Sum of Probabilities of 2 complementary events is unity.

Example 18: There are 12 teachers in maths department among them 2 are higher secondary teachers, 4 are secondary teachers and 6 are senior elementary teachers. If good performance award is to be given to only one teacher then find probability

a. That a secondary teacher will get the award

b. That a secondary teacher will not get the award

Solution: Here total number of teachers is 12. If this is assumed that all teachers are equally hardworking, dedicated and showing excellent performance throughout their career then the events will be equally likely.

a. Probability that a secondary teacher will get the award is $P(E) = \frac{4}{12} = \frac{1}{3}$

b. Probability that a secondary teacher will not get the award is $P(E') = 1 - P(E) = 1 - \frac{1}{3} = \frac{2}{3}$

11.3.3 Relative Frequency as an Estimate of Probability

When theoretical probability is not certain, we use relative frequency to estimate probabilities, as in most of the cases it is hard to predict the probability of an event happening finely.

Relative frequency is defined as the ratio between no. of times an event is occurring in an experimental trial and total number of trials in the experiment. i.e.

$$\text{Relative Frequency} = \frac{\text{frequency of an event occurring}}{\text{total no.of trials in an experiment}}$$

- The relative frequency is often calculated after performing experiment so it is often called experimental probability.
- More the no. of experiments done, closer the relative frequency becomes to the theoretical probability.
- Empirical probability is another name of relative frequency.
-

Example 19: A spinner of 5 equally likely colours red, blue, green, yellow and white is spun 50 times and it landed 10 times on red, 12 times on blue, 9 times on green, 9 times on yellow and 10 times on white colour. Find the relative frequency for the event that it lands on green colour.

Solution: Total No. of trials = n = 50

No. of favourable outcomes for landing on green colour = f = 9

$$\text{Relative Frequency} = \frac{\text{frequency of an event occurring}}{\text{total no.of trials in an experiment}} = \frac{9}{50}$$

Example 20: A fair coin is tossed 150 times and head comes up 70 times. What is the relative frequency of getting head up.

Solution: Total No. of trials = n = 150 No. of favourable outcomes for getting head up = f = 70

$$\text{Relative Frequency} = \frac{\text{frequency of an event occurring}}{\text{total no.of trials in an experiment}} = \frac{70}{150} = 0.466\ldots$$

11.3.4 Expected Frequency

In probability we often come across the situation where the probability of an event becomes slightly different from its theoretical probability. The expected frequency is defined as how often (we think) an event should happen on the basis of theoretical probability or on the basis of previously happened experiments (relative frequency).

For instance, If Ali rolled a fair die it is expected that 4 will happen1/6 times. If he rolls it 120 times, then he expects 4s to happen 120 x 1/6 = 20 times .Hence expected frequency is given by the formula : Expected Frequency = No. of trials x probability of the event = n x p

Example 21: 5 students out of a section of 30 students are expected to be absent everyday in summer camp. What No. of students is expected to be absent every day from 12 such sections of same strength.

Solution: Total no. of students in 12 sections = n = 12 30 = 360

$$\text{Probability of absent students} = p = \frac{5}{30} = \frac{1}{6}$$

$$\text{Expected frequency} = \text{No. of trials x probability of the event} = n \times p = 360 \times \frac{1}{6} = 60$$

Example 22: If Probability of a person to be Hafiz e Quran in a village is $\frac{3}{10}$, What do you expect the no. of Hafiz e Quran in 2500 such villages?

Solution: Total No. of villages = n = 2500 and Probability = p = $\frac{3}{10}$

Expected No. of Hafiz e Quran = No. of villages x probability of the event

$$= n \times p = 2500 \times \frac{3}{10} = 750$$

EXERCISE 11.3

1. A letter is chosen randomly from the word 'ALLAH'. Now find the probability of getting
   (i) a vowel (ii) an H (iii) an L (iv) a consonant

2. A garment factory shipped an order which contained 7000 jackets, 2000 sweaters, and 3000 trousers in which 7 trousers, 5 jackets and 9 sweaters are faulty. If the quality inspector unpacked one item at random, find the probability that it is
   (i) a trouser (ii) not a jacket (iii) a faulty item

3. A number wheel is divided into 8 equal sectors labelled as pansy, lily, orchid, tulip, jasmine, rose, marigold and sunflower. Maazz spins the wheel once. Find the probability that pointer:
   (i) Stops at rose (ii) stops at a 4 letter flower name
   (iii) Doesnot stop at marigold (iv) stops at a 3 letter flower name
   (vi) stops at a flower name

4. A decagonal die labelled 4, 4, 4, 4, 5, 5, 6, 7, 8, 8 is rolled once. Find the probability of getting:
   (i) a 4 (ii) an even number (iii) a multiple of 4
   (iv) not a 7 (v) an odd number (vi) a prime number
   (vii) LCM of 4 and 8 (viii) HCF of 4 and 8 (ix) factor of 12

5. A one digit whole number is chosen at random. Find the probability that it is:
   (i) less than 5 (ii) greater than 10
   (iii) not the largest 1 digit number (iv) additive identity of whole numbers
   (v) HCF of 3 and 5 (vi) multiplicative identity of real numbers
   (vii) not a prime number (viii) factor of 9

6. A coin is tossed 10 times. Complete the given table and answer the questions below the table.

Event	frequency	Relative frequency
head	6	
tail	4	

   (i) What is the expected frequency of getting head, if it was tossed 520 times?
   (ii) What is the expected frequency of getting tail, if it was tossed 305 times?

7. Complete the table and give answers, if a die is rolled 120 times

Number	Frequency	Relative frequency
1	15	...
2	25	...
3	15	...
4	20	...
5	30	...
6	15	...

(i) if it was rolled 6000 times, how many times occurrence of a 6 is expected?
(ii) if it was rolled 3000 times, how many times occurrence of a 1 is expected?
(iii) if it was rolled 400 times, how many times occurrence of a 2 is expected?

8. There are 6 girls sections of class 9 in a public school, namely Teal, orchid, mauve, hazel, zaffre and denim. Only one girl is to be chosen randomly from all sections for national science olympiad.
(a) Find the probability of selecting a girl from hazel section.
(b) Find the probability of not selecting a girl from hazel section.
(c) Verify that answers of both parts a and b add up to unity.
(d) If 6 girls are to be selected, find expected frequency of selecting from orchid section.
(e) If 60 girls are to be selected, what number of girls is expected to be chosen from Teal.

KEY POINTS

- Statistics is science of collecting and analyzing numerical data.
- Facts and statistics collected for analysis is called data.
- A table consisting of values of a data along with their frequencies is called a frequency table.
- In a group data, the table consisting of class intervals and their respective frequencies is called frequency table or a frequency distribution.
- The representation of data by a chart having adjacent rectangular bars is called histogram.
- The representation of data by a closed polygon in which class marks are plotted along x-axis and frequencies along y-axis is called a frequency polygon.
- A single value which can represent the whole data is called measure of central tendency, measure of location or an average.
- Measures of central tendency discussed in this grade are arithmetic mean, median and mode.
- Arithmetic mean is a ratio between the sum of items and number of items of the data.
- The middle most value of an arranged data is called median average or simply median.
- The most frequent value of the data is called modal average or mode.
- To find arithmetic mean of data whose values are not equally important, we use the weights instead of frequencies. This is called weighted arithmetic mean.

- The measure of chance of an event is called probability.
- The values of probability are from 0 to 1.
- The two events in which one shows non occurrence of other are called complementary events.
- Relative frequency is defined as the ratio between no. of times an event is occurring in an experimental trial and total number of trials in the experiment.
- The expected frequency of an event is the frequency we expect to see based on probabilities.

MISCELLANEOUS EXERCISE 11

1. Encircle the correct option in the following.

i. Which of the following is a class mark of the interval (10 − 15)?
(a) 10 (b) 12.5 (c) 15 (d) 16

ii. What is size of class interval (4 − 7)?
(a) 4 (b) 5 (c) 6 (d) 7

iii. Which of the following is chart of adjacent rectangles?
(a) bar graph (b) Frequency polygon (c) histogram (d) ogive

iv. Which of the following is measure of central tendency?
(a) variance (b) standard deviation (c) range (d) arithmetic mean

v. Which of the following is a formula of arithmetic mean for grouped data?
(a) $\frac{\sum x}{n}$ (b) $l+\frac{h}{f}(\frac{n}{2}-c)$ (c) $\frac{\sum fx}{\sum f}$ (d) $l+\frac{(f_m-f_1)\times h}{2f_m-f_1-f_2}$

vi. If arithmetic mean of 25 values is 10, then what is sum of values?
(a) 250 (b) 125 (c) 25 (d) 2.5

vii. What is median of the data 4, 3, 0, 2, 1?
(a) 0 (b) 2 (c) 3 (d) 4

viii. Which measure of central tendency can have more than one value?
(a) weighted mean (b) median (c) mode (d) arithmetic mean

ix. The probability of getting M in 'MUHAMMAD' is
(a) $\frac{1}{8}$ (b) $\frac{3}{8}$ (c) $\frac{3}{5}$ (d) none of these

x. Probability of picking an ace from a well shuffled pack of 52 playing cards is
(a) $\frac{1}{52}$ (b) $\frac{1}{13}$ (c) $\frac{4}{13}$ (d) none of these

xi. A normal fair die is rolled 6000 times. The expected number of 5 is
(a) 100 (b) 5000 (c) 1000 (d) none of these

xii. A fair coin is tossed 500 times. Expected number of tails is
(a) 100 (b) 250 (c) 1000 (d) none of these

xiii. in a group of 5 people, 4 like peach juice. The expected no. of people in a population of 1200 who like peach juice is
(a) 240 (b) 600 (c) 960 (d) none of these

xiv. How many times Haani should toss a fair coin if he expects to get 100 tails
(a) 100 (b) 200 (c) 1000 (d) none of these

xv. An event which can never happen is called
(a) sure event (b) certain event (c) possible event (d) impossible event

xvi. Two such events whose probabilities are ½ each are called
(a) likely (b) equally likely (c) unlikely (d) none of these

xvii. The probability of a certain event is
(a) 0 (b) $\frac{1}{2}$ (c) $\frac{3}{4}$ (d) 1

xviii. The probability of an event can take the value
(a) 0 (b) 1 (c) $\frac{1}{2}$ (d) all of these

xix. The probability of an event cannot take the value
(a) 0 (b) 1 (c) 2 (d) all of these

xx. The probability of an impossible event is
(a) 0 (b) 1 (c) 1/2 (d) all of these

2. For the following data, find mean, mode and median.

x	5	10	15	20	25	30	35	40
f	2	4	6	8	10	7	5	3

3. Find the median for the following frequency distribution.

Marks	30 – 39	40 – 49	50 – 59	60 – 69	70 – 79
Frequency	3	8	11	5	4

4. The arithmetic mean of 10 values is 35.5. If nine values are 20, 23, 37, 48, 29, 33, 45, 40, 45, find the tenth value

5. Find mean, median and mode for the following frequency distribution.

Classes	0 – 10	10 – 20	20 – 30	30 – 40	40 – 50
f	1	5	7	5	1

6. Affan is learning to play dart. He plays it 20 times each day and probability of hitting bull's eye is 1/10. Find:
(a) expected number of hits in 25 days. (b) expected number of misses in 25 days.

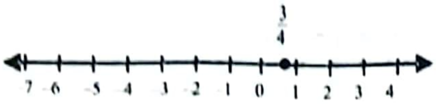
7. Shifaa usually gets late for her school 2days out of 6 working days of a week. What is the expected no. of days of her late arrival in 4 weeks.

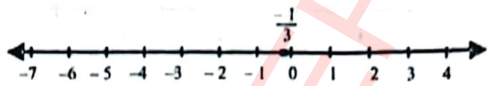
8. As'ha tries her best to score 100% in each maths assessment, but the probability of getting this is 0.8. Find the expected number of assessments in which she will score 100% out of total 20 assessments.

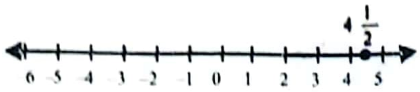
9. In a lab shelf there are 50 repoorts of covid 19. Among these 13 are females and 2 of the females are covid positive. If a report is picked at random, find:
a) The probability of getting a female report
b) The probability of getting a male report
c) The probability of getting a female covid negative report.

ANSWERS

Exercise 1.1

1. (i) 

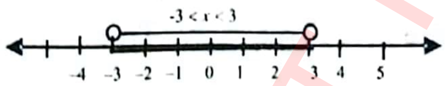
(ii) 

(iii) 

(iv) to (viii) Do yourself.

2. (i) multiplicative identity (ii) multiplicative inverse (iii) commutative property w.r.t. '+'
(iv) additive inverse (v) associative property w.r.t '+'
(vi) distributive property of multiplication over subtraction

3. (i) 

(ii) 

(iii) 

(iv) to (vi) Do yourself

4. (i) reflexive property (ii) transitive property (iii) symmetric property
(iv) multiplicative property (v) transitive property (vi) cancellation property
(vii) transitive property (viii) cancellation property

Exercise 1.2

1. (i) $6^{\frac{2}{3}}$ (ii) $\sqrt[5]{2^5}$, 2 (iii) $\sqrt[4]{x^4}$, x (iv) product rule is not applicable

(v) $\sqrt[4]{x^2}$, $x^{\frac{1}{2}}$ (vi) $\sqrt[3]{10^3}$, 10 (vii) $\sqrt[2]{10^2}$, 10 (viii) product rule is not applicable

2. (i) $\sqrt[3]{216^2}$, 36 (ii) $\sqrt{29}$ (iii) $\sqrt[5]{\frac{1}{32}}, \frac{1}{2}$ (iv) $\sqrt[3]{(216)^{-2}}, \frac{1}{36}$ (v) $\sqrt[3]{1000}$, 10 (vi) $\sqrt{39}$

3. (i) $5^{\frac{2}{3}}$ (ii) $10^2 = 100$ (iii) $-6^2 = -36$ (iv) $6^2 = 36$ (v) $-5^{\frac{2}{3}}$ (vi) $-10^2 = 100$

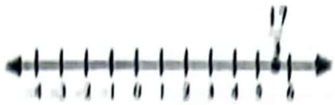
4. (i) 2^3 (ii) $7^{\frac{4}{3}} - 7$ (iii) $2^{\frac{13}{42}}$ (iv) $\frac{1}{\sqrt{3}}$ (v) 1 (vi) $\frac{3b^2}{a}$ (vii) 1

Exercise 1.3

1. 16,289
2. No
3. 43.5
4. 42 km
5. B
6. Rs. 77.7
7. 1
8. Capital = Rs.12,00,000 , debt = Rs. 8,00,000
9. F = -128.56 , K = 183.8
10. Farah= 76,144.83, Maryam = 152,289.66, Tehreem = 228,434.49

Miscellaneous Exercise 1

1. (i) (b) (ii) (d) (iii) (d) (iv) (c) (v) (a)
(vi) (d) (vii) (c) (viii) (c) (ix) (b) (x) (a)
(xi) (c)

2. (i)  (ii) 

(iii)  (iv) 

3. (i) $\sqrt[4]{16}$ (ii) $\sqrt[3]{(-3)^3}, -3$ (iii) $2^8, 128$ (iv) $(-2)^8, 256$ (v) $x^{-6}, \frac{1}{x^6}$

4. (i) $(-2)^3, -8$ (ii) $2^{\frac{1}{4}} . 3^1$

5. a. False b. False c. True c. True

Exercise 2.1

1. (i) 5.3407×10^{-4} (ii) 5.340×10^{7} (iii) 1.20×10^{-11} (iv) 2.5326×10^{0}
2. (i) 0.00009067 (ii) 5.64 (iii) 0.00000653 (iv) 3141500000
3. (i) 1.4318234×10^{4} (ii) 2300000 (iii) 1.2915783×10^{-2} (iv) 9988000
4. 1389 hrs app. & 58 days app. 5. 8.3535 minutes

Exercise 2.2

1. (i) no (ii) no (iii) yes (iv) no
2. (i) $6^3 = 216$ (ii) $\log_7 2401 = 4$ (iii) $5^5 = x$ (iv) $\log_b \frac{1}{27} = \frac{-3}{4}$ (v) $\log_{125} 25 = \frac{x}{3}$
(vi) $10^{12} = 10^{y}$ (vii) $\log_{256} \frac{1}{64} = \frac{x}{4}$ (viii) $x^3 + 1 = 3^2$ (ix) $(2x - 3) = 5^1$ (x) $\log_2 (2x + 1) = 3$
3. (i) $x = 3$ (ii) $x = 2$ (iii) $x = 4$ (iv) $x = 5$ (v) $x = 16$ (vi) $x = \pm 3$
4. (iii) 3125 (iv) 81 (v) 2 (vi) 12 (vii) -3
(viii) 2 (ix) 4 (x) 3.5

Exercise 2.3

1. 3.7253 2. 3.6609 3. 0.9828 4. 2.0449 5. 1.7192
6. -1.9183 or $\bar{2}.0817$ 7. -2.0315 or $\bar{3}.9685$ 8. 12.00 9. -4.3979 or $\bar{5}.6021$
10. 0.6020 11. 3.6020 12. 1.7324 13. 0.0039
14. $\bar{1}.003\,9 = -0.9961$ 15. impossible

Exercise 2.4

1. 270.6 2. 38.82 3. 1.748 4. 3661 5. 10000 6. 1.009
7. 0.1749 8. 0.08474 9. 0.003326 10. 1.000 11. 0.0010 12. 997700
13. 309.0 14. 3.090 15. 218.8

Exercise 2.5

1. (i) $2\log 3 + \log t$ (ii) $\log 59 - \log s$ (iii) $\log 5 + \log p + 2\log q - \log x - 3\log y$

(iv) $\frac{1}{2}(\log 53.3 - \log 46.4)$ (v) $2\log 5 + 5\log t + \frac{1}{3}\log a - \frac{1}{3}\log 4.4 - \frac{1}{3}\log t - 3\log b$

(vi) $\frac{2}{5}\log 7 + \frac{3}{5}\log t + \frac{1}{5}\log p - \frac{6}{5}\log d - \frac{2}{5}\log b$

(vii) $\frac{t}{3}\log 5.512 + t\log p + \frac{t}{2}\log m - \frac{t}{4}\log 5.91 - 2t\log a - t\log b$

2. (i) $\log \frac{x^3}{y^5}$ (ii) $\log \frac{\sqrt{t}\times\sqrt[3]{r}}{\sqrt[5]{s}}$ (iii) $\log \sqrt[7]{\frac{57.7\times(36.6)^4}{(9.24)^3\times(23.3)^2}}$

(iv) $\log \frac{6^5 t^{\frac{1}{3}}\times\sqrt[3]{a^2}}{\sqrt{32.2}\times(9.42)^7}$ (v) $\log \sqrt[4]{\frac{(37.74)^5\times 28.83}{53.71}}$

3. (i) 3.9069 (ii) $\frac{1}{3}$ (iii) 3.7997 (iv) 7.7942 (v) 6

4. (i) 0.0791 (ii) 1.0458 (iii) 0.1688 (iv) -1.6198 or $\bar{2}.380$ (v) 0.1046

Exercise 2.6

1. (i) 15 (ii) 201 (iii) 4 (iv) 26 (v) 82 (vi) 21

2. (i) 140.6 (ii) 8.914 (iii) 6.209 (iv) 0.5664 (v) 0.6647 (vi) 0.5631
(vii) 1.090 (viii) 0.03301

3. app. 5 times

Miscellaneous Exercise 2

1. (i) (d) (ii) (c) (iii) (d) (iv) (a) (v) (a) (vi) (b)
(vii) (d) (viii) (c) (ix) (b) (x) (d) (xi) (b) (xii) (b)
(xiii) (b) (xiv) (b) (xv) (c) (xvi) (a) (xvii) (c) (xviii) (c)

2. (i) 5.336×10^{1} (ii) 1.02×10^{-14} (iii) 5.234×10^{-1}

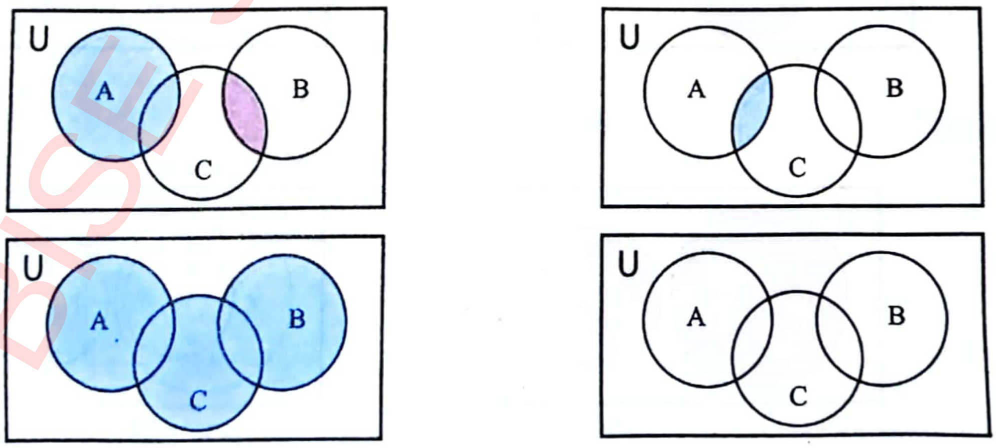
3. (i) 0.07232 (ii) 21386.43 (iii) 5.6 **4.** (i) 0 (ii) 2 (iii) 1

5. (i) 2 (ii) 16 (iii) ± 2 (iv) 8

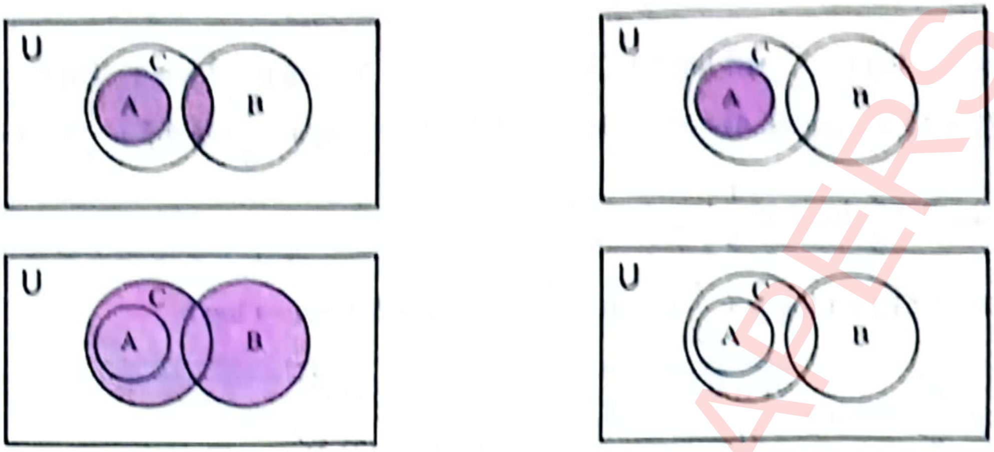
6. (i) 1.4771 (ii) $\bar{1}.3802$ (iii) 2.5563 **7.** 31626957.07

Exercise 3.1

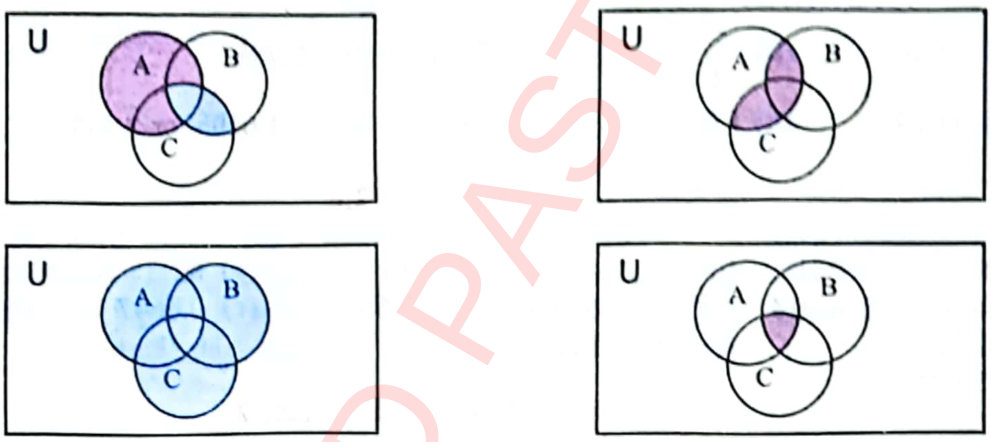
1. (i)



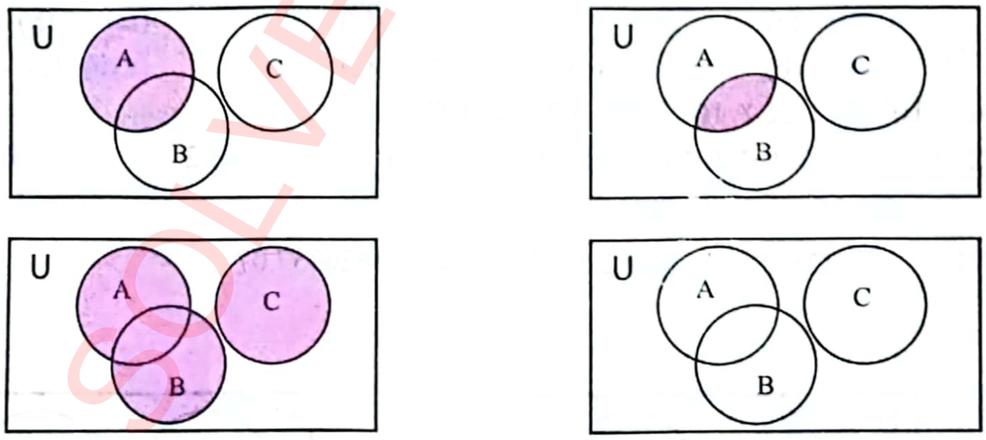
(ii)



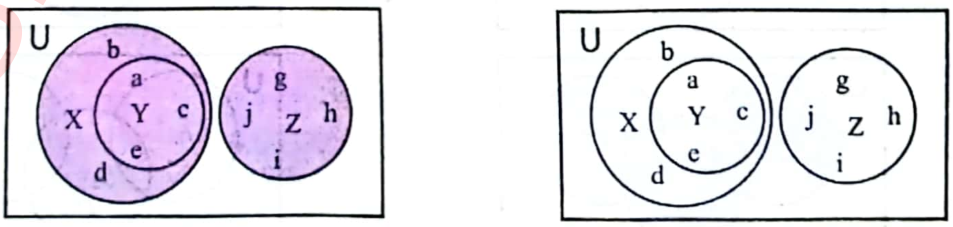
(iii)



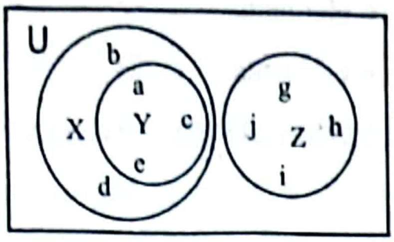
(iv)



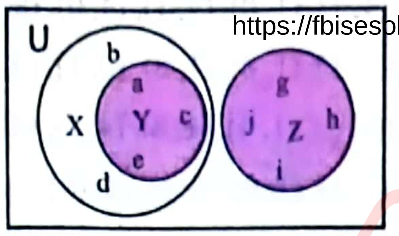
2. (i) & (ii) (iii) & (iv)



(v)



(vi)

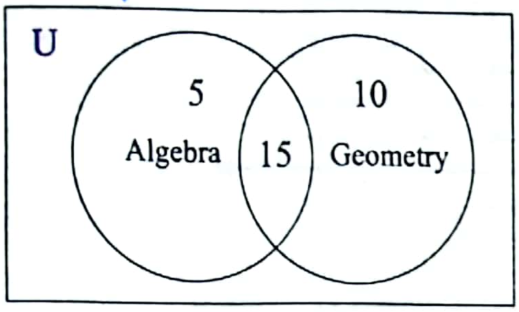


Exercise 3.2

1. 11

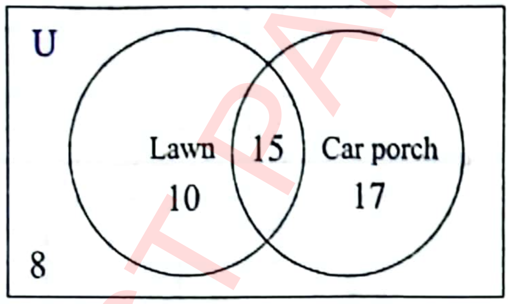
2. n(B – A) = 19, n(B) = 21

3.



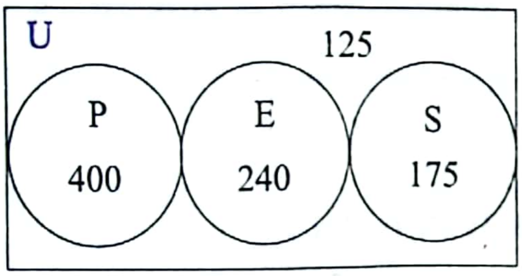
25 students like geometry.

4.



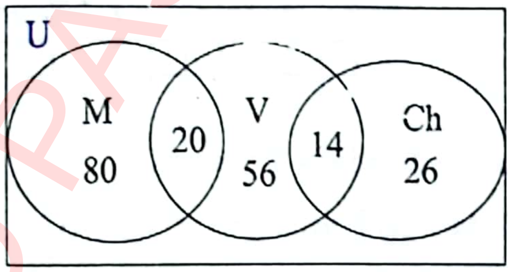
Houses without lawn & porch: 8

5.



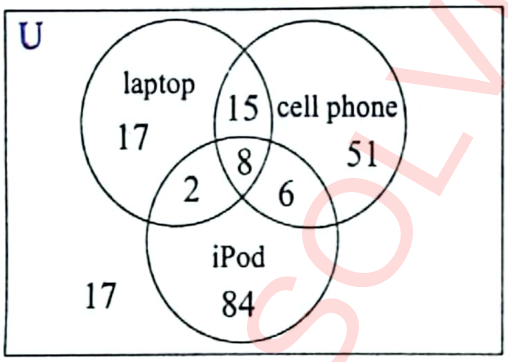
Out of school children: 125

6.



only mango: 80, only vanilla: 56
only chocolate: 26

7.



(i) 51
(ii) 17
(iii) 103

8. 480 9. 32 10. 18 11. 96

12. Arabic only: 40, French only: 100, Both: 152

13. (i) 210 (ii) 20 (iii) 380 14. 38, 22

Exercise 3.3

1. (i) $a = 7, b = -1$ (ii) $a = -5, b = -1$ (iii) $a = 6, b = -5$ (iv) $x = -14, y = 8$

(v) $x = -6, y = -16$ (vi) $x = 1, y = \frac{1}{3}$ (vii) $x = \frac{5}{6}, y = \frac{11}{6}$

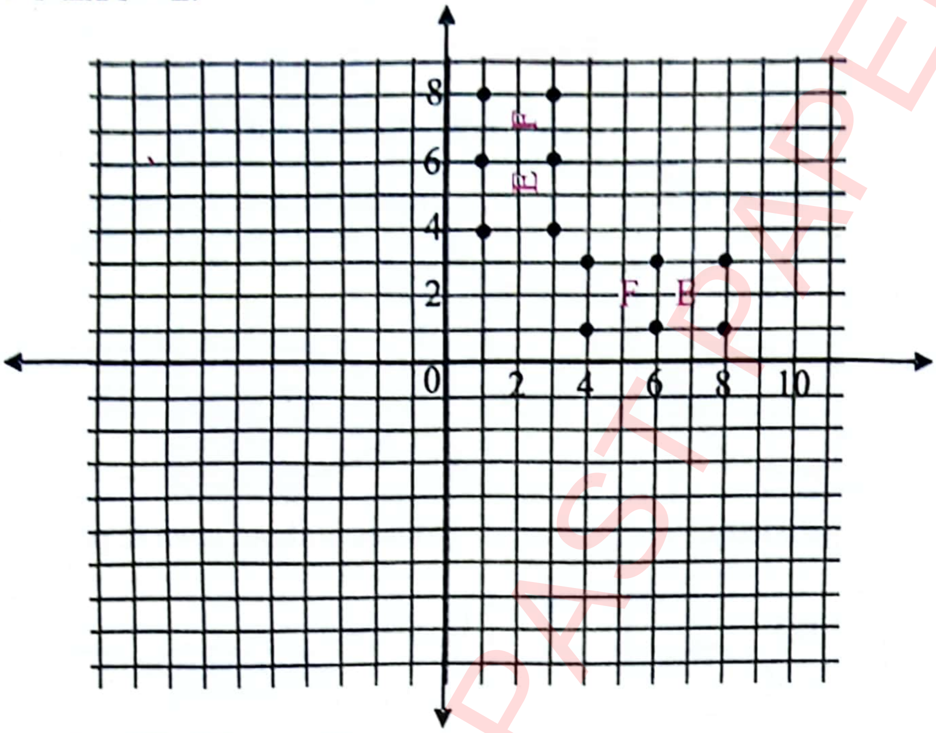
2. (i) $A \times B = \{(1, 1), (1, 0), (4, 1), (4, 0), (8, 1), (8, 0)\}$ 6 elements

(ii) $B \times A = \{(1, 1), (1, 4), (1, 8), (0, 1), (0, 4), (0, 8)\}$ 6 elements

(iii) $A \times A = \{(1, 1), (1, 4), (1, 8), (4, 1), (4, 4), (4, 8), (8, 1), (8, 4), (8, 8)\}$ 9 elements

(iv) $B \times B = \{(1, 1), (1, 0), (0, 1), (0, 0)\}$ 4 elements

3. Graph of $E \times F$ and $F \times E$:



Draw graphs of $E \times E$ and $F \times F$ yourself.

4. $L = \{0, 1\}$, $M = \{2, 3, 4\}$, $M \times L = \{(2, 0), (2, 1), (3, 0), (3, 1), (4, 0), (4, 1)\}$

5. (i) $\{(1, 2), (1, 4), (1, 6), (1, 7), (3, 2), (3, 4), (3, 6), (3, 7), (5, 2), (5, 4), (5, 6), (5, 7)\}$

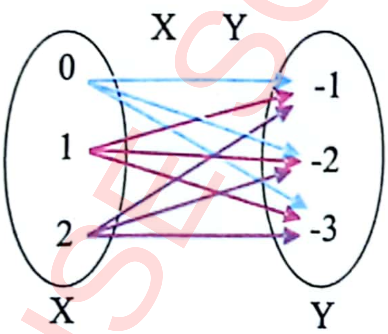
(ii) $\{(1, 2), (1, 4), (3, 2), (3, 4), (5, 2), (5, 4), (1, 6), (1, 7), (3, 6), (3, 7), (5, 6), (5, 7)\}$

(iii) Verified from (i) and (ii).

6. (i) $\{(a, c), (e, c), (i, c)\}$ (ii) $\{(a, c), (e, c), (i, c)\}$ (iii) Verified from (i) and (ii).

7. (i) LHS = RHS = $\{(1, 1), (2, 1)\}$ (ii) Verify yourself.

8.



Draw arrow diagram for $X \times Y$ yourself.

Exercise 3.4

1. (i) $2^6 = 64$ (ii) 2^{49} (iii) 2^9

2. (i) 8, $R_1 = \{\ \}$, $R_2 = \{(\sqrt{2}, \sqrt[3]{5})\}$, $R_3 = \{(\sqrt{3}, \sqrt[3]{5})\}$, $R_4 = \{(\sqrt{5}, \sqrt[3]{5})\}$,

$R_5 = \{(\sqrt{2}, \sqrt[3]{5}), (\sqrt{3}, \sqrt[3]{5})\}$, $R_6 = \{(\sqrt{2}, \sqrt[3]{5}), (\sqrt{5}, \sqrt[3]{5})\}$, $R_7 = \{(\sqrt{3}, \sqrt[3]{5}), (\sqrt{5}, \sqrt[3]{5})\}$,

$R_8 = \{(\sqrt{2}, \sqrt[3]{5}), (\sqrt{3}, \sqrt[3]{5}), (\sqrt{5}, \sqrt[3]{5})\}$.

(ii) 16, $R_1 = \{\ \}$, $R_2 = \{(\pi,\pi)\}$, $R_3 = \{(\pi,e)\}$, $R_4 = \{(e,\pi)\}$, $R_5 = \{(e,e)\}$,

$R_6 = \{(\pi,\pi),(\pi,e)\}$, $R_7 = \{(\pi,\pi),(e,\pi)\}$, $R_8 = \{(\pi,\pi),(e,e)\}$, $R_9 = \{(\pi,e),(e,\pi)\}$,

$R_{10} = \{(\pi,e),(e,e)\}$, $R_{11} = \{(e,\pi),(e,e)\}$, $R_{12} = \{(\pi,\pi),(\pi,e),(e,\pi)\}$,

$R_{13} = \{(\pi,\pi),(\pi,e),(e,e)\}$, $R_{14} = \{(\pi,\pi),(e,e),(e,\pi)\}$,

$R_{15} = \{(\pi,e),(e,e),(e,\pi)\}$, $R_{16} = \{(\pi,\pi),(\pi,e),(e,\pi),(e,e)\}$

(iii) 4, $R_1 = \{\ \}$, $R_2 = \{(5,1)\}$, $R_3 = \{(5,10)\}$, $R_4 = \{(5,1),(5,10)\}$. (iv) Do yourself.

3. $R_1 = \{(7,7),(7,8)\}$, Dom $R_1 = \{7\}$, Rang $R_1 = \{7,8\}$

$R_2 = \{(7,9),(8,7),(8,9)\}$, Dom $R_2 = \{7,8\}$, Rang $R_2 = \{7,9\}$

4. (i) $\{(5,5),(7,7),(9,9)\}$

(ii) $\{(5,7),(5,9),(5,11),(6,7),(6,9),(6,11),(7,9),(7,11),(8,9),(8,11),(9,11)\}$

(iii) $\{(6,5),(7,5),(8,5),(8,7),(9,5),(9,7)\}$ (iv) $\{(6,7),(8,9)\}$

(v) $\{(6,5),(8,7)\}$ (vi) $\{(5,7),(7,9),(9,11)\}$

5. (i) $R = \{(2,4),(2,6),(2,8),(2,12),(4,4),(4,8),(4,12),(6,12)\}$

(ii) Dom $R = \{2,4,6\}$, Rang $R = \{4,6,8,12\}$

(iii) $R^{-1} = \{(4,2),(6,2),(8,2),(12,2),(4,4),(8,4),(12,4),(12,6)\}$

(iv) Do yourself.

6. (i) $R = \{(x,y) / x-y=2\}$ (ii) Dom $R = \{2,4,6,8,10\}$, Rang $R = \{0,2,4,6,8\}$

(iii) $R^{-1} = \{(0,2),(2,4),(4,6),(6,8),(8,10)\} = \{(y,x) / y-x=2\}$ (iv) Do yourself.

7. $R = \{(0,1),(1,3),(3,7)\}$, $R^{-1} = \{(1,0),(3,1),(7,3)\}$

8. (i) $R_1 = \{(1,3^0)\}$, Dom $R_1 = \{1\}$, Rang $R_1 = \{3^0\}$

(ii) $R_2 = \{(2,3^0),(4,3^0),(4,3^1),(8,3^0),(8,3^1)\}$, Dom $R_2 = \{2,4,8\}$, Rang $R_2 = \{3^0,3^1\}$

(iii) $R_3 = \{(1,3^0),(1,3^1),(1,3^2)\}$, Dom $R_3 = \{1\}$, Rang $R_3 = \{3^0,3^1,3^2\}$

(iv) $R_4 = \{(1,3^0),(1,3^1),(1,3^2)\}$, Dom $R_4 = \{1\}$, Rang $R_4 = \{3^0,3^1,3^2\}$

(v) $R_5 = \{(1,3^1),(1,3^2),(2,3^2),(4,3^2)\}$, Dom $R_5 = \{1,2,4\}$, Rang $R_5 = \{3^1,3^2\}$

Miscellaneous Exercise 3

1. (i) b (ii) c (iii) c (iv) a (v) b (vi) d (vii) b (viii) c

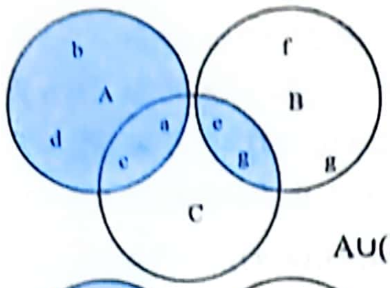
(ix) a (x) d (xi) d

2. (i) 2^{16} (ii) $R_1 = \{(a,b)\}$, $R_2 = \{(a,e),(f,j)\}$, $R_3 = \{(a,j),(f,n),(f,e)\}$

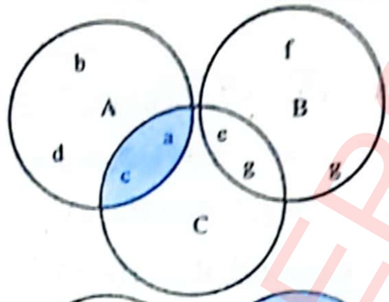
(iii) $\{(a,b),(f,b),(h,e),(s,n),(s,j)\}$ (iv) $\{(a,b),(f,e),(h,j),(s,n)\}$

(v) $\{(a,b),(f,j),(h,e),(s,n)\}$

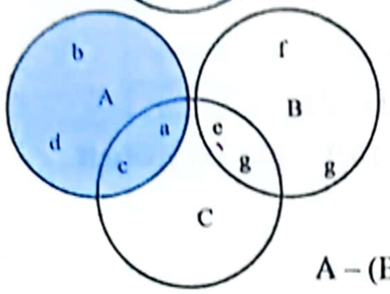
3.



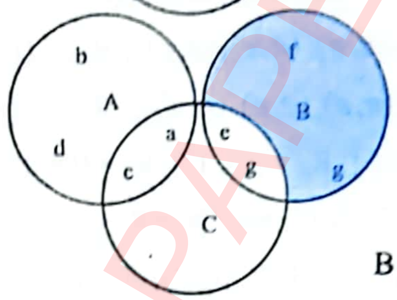
A∪(B∩C)



A∩(B∪C)

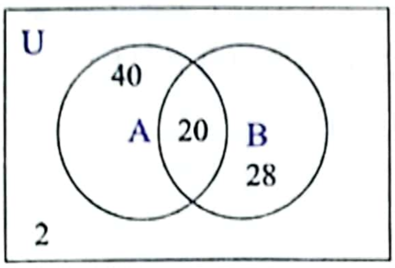


A – (B∩C)

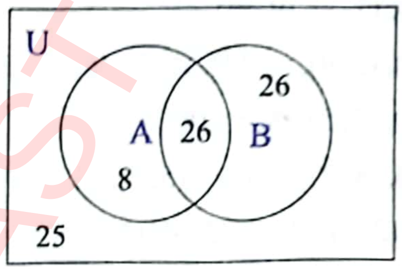


B – (A∪C)

7. (i)



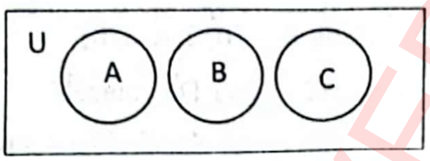
(ii)



8. (i) A = {Haani, Zubair, Haider}, B = {Abdullah, Umer , Bilal, Ali},

C = {Hassan, Jaffer, Usman}

(ii)



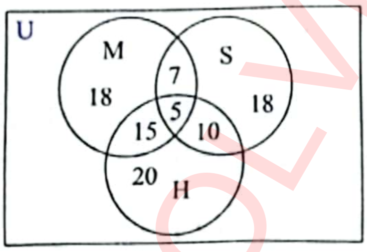
(iii) $A \cap (B \cap C)^c$ = {Haani, Zubair, Haider}

$(A \cup B) \cap C^c$ = {Haani, Zubair, Haider, Abdullah, Umer , Bilal, Ali}

$A - (B \cap C)$ = {Haani, Zubair, Haider}

$A - (A \cup B)$ = { }

9. (i)



(ii) 93

(iii) 7

Exercise 4.1

1. $2xy^2(xy - 3x + y)$
2. $3(n - 1)(x - y)$
3. $18(x^2 + 3y^2)^2$
4. $(k - 2)^2$
5. $(3x + 13y - 2)(3x - 13y - 2)$
6. $-2(x + 1)(x - 1)$
7. $(x - 3a + 4b)(x - 3a - 4b)$
8. $(1 + x + y)(1 - x - y)$
9. $x^2 + y^2 + 2xy - 4c^2$
10. $(x + 2y)^2 - (z)^2$
11. (a) $2y - 9$ (b) $5x - 3$
12. $(x^2 + 2m^2 + 2xm)(x^2 + 2m^2 - 2xm)$
13. $(m^2 + 1 + m)(m^2 + 1 - m)$
14. $-3x^2(x - 1)(x + 8)$
15. $(x^4 - x^2 + 1)(x^2 + x + 1)(x^2 - x + 1)$
16. $4(x^2 + 8y^2 + 4xy)(x^2 + 8y^2 - 4xy)$
17. $(x - 3)(x - 4)$
18. $(x - 1)(x - 8)$
19. $(2z - 5)(5z - 2)$
20. $(3y - 1)(4 - y)$
21. $(x - 6)(x - 15)$
22. $(x - 1)(x + 2)$
23. $(x + 3)(3x + 2)$
24. $(2x + y)(x - 3y)$
25. $(4 + 5x)(2 - x)$
26. $(2 + x)(3 - 5x)$
27. $2(a + 1)(a - 3)$
28. $(u + 2)(u - 2)(u + 3)(u - 3)$
29. $(y - 4)(y + 4)(y^2 + 4)$

Exercise 4.2

1. $(x-5)(x^2+5x+25)$
2. $(2x+1)(4x^2-2x+1)$
3. $3(pq-3x)(p^2q^2+3pqx+9x^2)$
4. $(3+8x)(9-24x+64x^2)$
5. $(t+2)(t-2)(t^2-2t+4)(t^2+2t+4)$
6. $(x^2+y^2)(x^4-x^2y^2+y^4)$
7. $(y-x)(x^2+y^2-6x-6y+xy+12)$
8. $(4x+4y-z)(16x^2+16y^2+32xy+4xz+4yz+z^2)$
9. $(3p-4q)^3$
10. $(2p+q)^3$
11. $(5x-y)^3$
12. $(p-3q)^3$
13. $(2x^2-3x+11)(x+1)(2x-5)$
14. $(y^2+2y+3)(y^2+2y+5)$
15. $(y^2-4y+1)(y^2-4y+2)$
16. $(k^2+7k-5)(k^2+7k-15)$
17. $(x^2-3x-12)(x^2-3x-16)$
18. $(x^2-8x-6)(x^2+2x-6)$
19. $(x^2-9x+12)(x^2-6x+12)$
20. $(10+10x-x^2)(10+2x-x^2)$
21. $(a^2+9)(a^4-9a^2+81)$, sum of two cubes because of positive sign between the terms.
22. $(t+2)(t+2)(t+2)$, each factor is a binomial.

Exercise 4.3

1(a).

1	p	q	–	pq
5	p^2	q	r	$5p^2qr$
7	P^2	q	r^2	$7p^2qr^2$

b. $12pq(p+q)^2$ c. $(x+y)^2$, $(x-y)^2$ etc. d. $m-n$, $(m-n)(m^2+n^2)$ 2. $x+y$ 3. $a^2-2ab+b^2$

4. $ab(a-b)$ 5. $x-7$ 6. $3x+1$ 7. $cx+d$ 8. m^2-n^2 9. $a(x+a)$

Exercise 4.4

1.(i) x^2+2x (ii) x^2+2x (iii) x^2+2x (iv) $5(x^2+2x)$ (v) No, HCF is x^3+2x^2

2.

(i)	8	x^6	y	z	$8x^6yz$
(ii)	24	x^3	y^4	z	$24x^3y^4z$
(iii)	18	x^6	y^2	z^3	$18x^6y^2z^3$
(iv)	28	x^3	y^5	z^5	$28x^3y^5z^5$

3. $a+2$ x^2+4x+4 5. $x-1$ 6. p^2+1 7. $x(3x+2)$
8. $x+3$ 9. $2ab(9a^2-1)$ 10. $p^2q^2(p+q)(p^3-q^3)$ 11. $xy(2x+1)(2x-1)$
12. $(x-3)(x-1)(x+2)$ 13. $(m^2+1)(m^6-1)$ 14. x^4-5x^2+4 15. $(x+5)(x-4)(x-6)$

Exercise 4.5

1. (i) $x-2$, $-4(x^2-4)(x+3)$ (ii) HCF: a^2+a+1, LCM: $(a^3-1)(a^2-a+1)(a-1)$
(iii) $x(x+3)$, $x(x+3)(x-1)(2x-1)$
2. $x^4-17x^3+81x^2-7x-490$ 3. $x^3+5x^2-2x-24$ 4. $y+4$
5. x^4+x^3-x-1 6. $(3x^3+x^2+x-2)(4x^3-10x^2+4x+2)$

Exercise 4.6

1.(i) $\dfrac{1}{x^3+7x-8}$ (ii) $\dfrac{x^2}{x^2+y^2}$ (iii) $\dfrac{n}{m^2+n^2}$ (iv) 1

2. $\dfrac{7}{12}$ 3. $\dfrac{ab}{2a-1}$ 4. 1 5. $\dfrac{2xy}{x-y}$ 6. $\dfrac{4a^2+2a+1}{6a}$ 7. $\dfrac{x-5}{x-1}$

8. $\frac{x+2y}{4x^2-9y^2}$ 9. $\frac{x^2+y^2}{xy(x^2-y^2)}$ 10. $\frac{4}{3}$ 11. $\frac{12a^2-4a+7}{3(4a^2-9)}$ 12. $\frac{23x}{(1+2x)(2+x)(5-9x)}$ 13. $\frac{1-q}{p}$

Exercise 4.7

1. $\pm(4x-7)$ 2. $\pm a(5a-3)$ 3. $\pm(x^2-\frac{1}{x^2}+2)$ 4. $\pm(a-\frac{1}{a}-4)$
5. $\pm(a^2+10a+20)$ 6. $\pm(x^2+4x+2)$ 7. $\pm(x^2+5x+3)$ 8. $\pm(7b^2+2ab+a^2)$
9. $\pm(2x^2-3x+5)$ 10. $\pm(1-5x+x^2)$
11. $\pm(x^2+2-\frac{1}{x^2})$ 12. $\pm\left(a-4-\frac{7}{a}\right)$ 13. $\pm(x^2-x+\frac{1}{4})$ 14. $\pm(2x^2+8+\frac{8}{x^2})$
15. (i) $-a-1$ (ii) $a+1$ (iii) $a=-1$ 16. $p=216$, $q=324$ 17. $k=8$

Exercise 4.8

1. a) $(x-3)$m, $(x+1)$m b) P= $(4x-4)$m c) Rs. $(80x-80)$ d) Rs. $250(x^2-2x-3)$
2. a) $(5x-3)$m b) P $(20x-12)$m c) Rs. 75 d) Rs.$75(25x^2-30x+9)$
3. a) $(5x-2)$m b) $6(5x-2)^2$ c) Rs. $192(5x-2)^2$

4. radius 1=x-3, radius 2=x-5 5. 5m-6n, 5m+6n 6. if speed = x+2 then distance = x+3 and vice versa

Miscellaneous Exercise 4

1. (i) (d) (ii) (d) (iii) (c) (iv) (a) (v) (a) (vi) (c) (vii) (d)
(viii) (c) (ix) (d) (x) (a) (xi) (b) (xii) (b) (xiii) (b) (xiv) (b)
(xv) (d) (xvi) (a) (xvii) (c) (xviii) (c) (xix) (a) (xx) (c)

2. $(a+3)(a-3)(a^2-14)$ 3. $(2x+3)(3x+5)$ 4. $(x^2-8x-7)(x^2-8x+13)$
5. $(x^2+2x-4)(x^2-2x-4)$ 6. $(3x^2+4x-1)(3x^2+4x-6)$ 7. $(m+n)(m-n)(2m^2+3n^2)$
8. $(x+2y)(x-2y)(x^2+14y^2)$ 9. $(x+2a)(x-b)$
10. $(a+b)(a-b)(a^2+b^2)(a^2+ab+b^2)(a^2-ab+b^2)(a^4-a^2b^2+b^4)$
11. $4x^2y(x+3)(7x-5)$ 12. $(x-y)(2x-2y+1)(2x-2y-1)$
13. $(p+2q)(p-2q)(x-2y)(x^2+2xy+4y^2)$ 14. $(x^2+y^2+3xy)(x^2+y^2-3xy)$

15. (i) $x+3$ (ii) $3x^2+1$
16. (i) $(x+1)(x+2)(2x+1)$ (ii) $(x+2)(x+3)(3x+2)$
17. $a+b+c$, $(a+b+c)(a-b)(b-c)(c-a)$
18. $\pm(\frac{3a}{x}-\frac{1}{5}+\frac{2x}{3a})$ 19. (i) x (ii) $\frac{x+2}{(x+1)(x+3)}$ 20. $a=-2$, $b=5$

Exercise 5.1

1. $\{-3\}$ 2. $\{1\}$ 3. $\{1\}$ 4. $\left\{\frac{-2}{5}\right\}$ 5. $\{8\}$ 6. $\{7\}$ 7. $\{10\}$ 8. $\left\{-\frac{1}{4}\right\}$
9. $\{24\}$ 10. $\left\{-\frac{3}{2}\right\}$ 11. $\{11\}$ 12. $\{9\}$ 13. $\{-1.3\}$

Exercise 5.2

1. {8} 2. {7} 3. {14} 4. {40} 5. {40} 6. ϕ 7. {21}
8. {13} 9. {3} 10. {3} 11. {3} 12. ϕ 13. {64} 14. {10} 15. {6} 16. {–3}

Exercise 5.3

1. $\left\{\frac{5}{3}, -\frac{5}{3}\right\}$ 2. $\{4, -8\}$ 3. $\{2, -\frac{8}{5}\}$ 4. $\{1, -3\}$ 5. $\{2\}$ 6. $\left\{\frac{4}{3}, \frac{8}{3}\right\}$
7. $\{3, -2\}$ 8. $\{-3, \frac{5}{3}\}$ 9. $\{9, -9\}$ 10. ϕ 11. $\left\{\frac{13}{2}, -\frac{11}{2}\right\}$ 12. $\left\{5, -\frac{1}{7}\right\}$
13. $\left\{\frac{5}{2}, -\frac{2}{3}\right\}$ 14. $\{1, -7\}$

Exercise 5.4

1(a). (i) Yes (ii) No (iii) Yes (iv) Yes (v) Yes (b) (i) {2, 3, 4, 5} (ii) {1, 2, 3, 4, 5, 6}
(iii) $\{z \mid z \in R \wedge z \leq -3\}$ (iv) {0, 1, 2, 3, 4} (v) $(x \mid x \in R \wedge -4 < x < -\frac{3}{2}\}$
2. {1, 2} 3. {0, 1, 2, 3, 4, 5} 4. $\{y \mid y \in R \wedge y > 3\}$ 5. $\{y \mid y \in R \wedge y \leq 6\}$
6. $\{x \mid x \in R \wedge x > -2 \text{ or } x < -6\}$ 7. $\{x \mid x \in R \wedge -1 < x \leq 3\}$
8. $\{x \mid x \in R \wedge -5 \leq x \leq 2\}$ 9. $\{x \mid x \in R \wedge x < -5 \text{ or } x > -1\}$
10. $\{x \mid x \in R \wedge -4 \leq x < -3\}$ 11. $\{x \mid x \in R \wedge x < -9 \text{ or } x > -3\}$
12. $\{x \mid x \in R \wedge x \geq 13\}$

Miscellaneous Exercise 5

1. (i) (d) (ii) (a) (iii) (c) (iv) (c) (v) (b) (vi) (c)
(vii) (d) (viii) (c) (ix) (a) (x) (b) (xi) (b) (xii) (b)
(xiii) (c) (xiv) (d) (xv) (b) (xvi) (b)
2. $x = 10$ 3. $a = \frac{3}{2}$ 4. ϕ 5. $\left\{\frac{13}{5}, -3\right\}$ 6. $\{x \mid x \in R \wedge x \leq -8\}$
7. $\{y \mid y \in R \wedge y > \frac{-3}{2}\}$ 8. $\{x/x \in R \wedge x \leq -2 \text{ or } x \geq 6\}$ 9. $\{y/y \in R \wedge y < 10 \text{ or } y > 19\}$

Exercise 6.1

1. (i) 18,000" (ii) 18,00" (iii) 37,800" (iv) 73,220"
2. (i) 4,500' (ii) 2' (iii) 3,040' (iv) 1,830.5'
3. (i) 2.25° (ii) 0.0417° (iii) 61° (iv) 45.7625°
4. (i) 60°7'30" (ii) 0°2'15" (iii) 1°0'51" (iv) 255°27'

5. (i) $\frac{\pi}{4}$ rad (ii) $\frac{5\pi}{6}$ rad (iii) $\frac{121\pi}{360}$ rad (iv) $\frac{2\pi}{3}$ rad
6. (i) 4.71 rad (ii) 1.05 rad (iii) 3.15 rad (iv) 1.32 rad
7. (i) 45° (ii) 150° (iii) 15° (iv) 31°30'
(v) 57°17'45" (vi) 180° (vii) 2160° (viii) 286°28'44"

Exercise 6.2

1. (i) $5.24cm, 13.09cm^2$ (ii) $25.13m, 150.77m^2$ (iii) $6.36dm, 19.08dm^2$
2. $5.24cm, 10.48cm^2$
3. (i) 2 rad (ii) 3.78 rad (iii) 0.79 rad (iv) 0.5 rad
4. (i) $1.27cm$ (ii) $22.9m$ (iii) $22.57cm$ (iv) $20dm$
5. $15.71 inch$ 6. 84° 7. $30.85cm, 56.55cm^2$
8. (i) $\frac{\pi}{2}$ (ii) $\frac{7\pi}{12}$ (iii) $\frac{3\pi}{8}$

Exercise 6.3

1. Complete yourself.
2. (i) 0 (ii) $\frac{7}{4}$ (iii) $\frac{1}{\sqrt{3}}$ (iv) $-\frac{1}{2}$ (v) $\frac{1}{2}$
4. (i) 45° (ii) 270° (iii) 45° (iv) 30° (v) 90°
5. $\sin\theta = \frac{\sqrt{3}}{2}$, $\tan\theta = \sqrt{3}$, $\cot\theta = \frac{1}{\sqrt{3}}$, $\sec\theta = 2$, $\operatorname{cosec}\theta = \frac{2}{\sqrt{3}}$ 6. $\frac{\sqrt{3}}{4}$
7. $\sin\theta = -\frac{1}{\sqrt{2}}$, $\cos\theta = \frac{1}{\sqrt{2}}$, $\cot\theta = -1$, $\sec\theta = \sqrt{2}$, $\operatorname{cosec}\theta = -\sqrt{2}$

Exercise 6.4

3. (i) 30° (ii) 45° (iii) 45°
4. (i) $\angle A = \angle B = 45°$, $b = 4cm$ (ii) $\angle R = 30°$, $QR = 12cm$, $PR = 8\sqrt{3}\,cm$
(iii) $\angle P = \angle Q = 45°$, $PQ = 8\sqrt{2}\,m$ (iv) $\angle Y = 30°$, $\angle Z = 60°$, $XY = 8\sqrt{3}\,m$
(v) $\angle N = 90°$, $\angle L = 36.87°$, $\angle M = 53.13°$ 5. r^2

Exercise 6.5

1. 11.55 m 2. 30° 3. 1.39 m 4. 10 m
5. 52 m 6. 50.7 m 7. 34.7 m 8. 18.93 m, 12 m
9. 80m , 33.69° 10. 0.73m/s

Exercise 6.6

1. measure yourself 2. find yourself 3. 070° 4. 7.07km , 270°
5. 4km, 103° 6. 15.6 m 7. 13km 8. 9.96km , 8.36km

Miscellaneous Exercise 6

1. (i) (a) (ii) (b) (iii) (d) (iv) (c) (v) (b) (vi) (a) (vii) (c) (viii) (d) (ix) (d) (x) (b) (xi) (c) (xii) (a) (xiii) (c) (xiv) (c) (xv) (d)
2. $\sin\alpha = \frac{12}{13}$, $\tan\alpha = \frac{-12}{5}$, $\sec\alpha = \frac{-13}{5}$,
3. $5.77\ m/s$
4. 2.69
5. $10cm$, $10cm$, $10\sqrt{2}cm$
6. 28.39 m
7. 163 m
8. 050°

Exercise 7.1

1. (i) 5 (ii) 10 (iii) $2\sqrt{10}$ (iv) $10\sqrt{2}$ (v) $\sqrt{5}$
   (vi) $\sqrt{17}$ (vii) $\frac{\sqrt{29}}{2}$ (viii) $\sqrt{\frac{193}{2}}$ (ix) $\frac{5\sqrt{10}}{12}$ (x) $\frac{5\sqrt{29}}{4}$
2. (i) non-collinear (ii) non-collinear (iii) collinear (iv) collinear (v) collinear
   (vi) non-collinear
3. (i) right triangle; perimeter = $5+\sqrt{13}$ units
   (ii) equilateral triangle; perimeter = 6 units
   (iii) scalene triangle; perimeter $\approx$ 24.49 units
   (iv) right triangle; perimeter $\approx$ 27.21 units
   (v) isosceles triangle; perimeter = $6 + 2\sqrt{34}$ units
   (vi) right isosceles triangle; perimeter = $12 + 6\sqrt{2}$ units
8. Yes

Exercise 7.2

1. (i) (4, 7) (ii) $\left(\frac{1}{2}, 1\right)$ (iii) (2, 1)
2. (i) $E(3, 1)$; $F\left(\frac{7}{2}, 4\right)$; $G\left(-\frac{1}{2}, 4\right)$; $H(-1, 1)$ (ii) Draw yourself.
   (iii) $EF = GH = \frac{\sqrt{37}}{2}$ units; FG = HE = 4 units (iv) rectangle
3. Diagonals do not bisect each other.
4. Length of diagonals are $4\sqrt{2}$ units, $2\sqrt{2}$ units.
5. (i) D(−0.5, 3.5) ; E(3.5, 3.5) (ii) DE = 4 units ; BC = 8 units
   (iii) $\frac{1}{2}BC = 4$ units (iv) $DE = \frac{1}{2}BC$
6. (i) Hassan's starting point is (6, 0) and Ali's starting point is (0, 6) (ii) $C(3\sqrt{2}, 3\sqrt{2})$
7. (i) M(3, 3) ; N(0, 3) ; O(0, 0) (ii) A(0,0) ; C(−4, 2)

Miscellaneous Exercise 7

1.

(i) b	(ii) a	(iii) a	(iv) c	(v) c	(vi) d
(vii) c	(viii) b	(ix) d	(x) c	(xi) b	(xii) a
(xiii) b	(xiv) a	(xv) c	(xvi) d	(xvii) d	(xviii) b
(xix) c	(xx) c	(xxi) c	(xxii) b	(xxiii) d	(xxiv) a

Exercise 8.1

1. (i) 0 (ii) 0.577 (iii) 1.732 (iv) undefined (v) – 1.732 (vi) – 0.577
   (vii) – 0.176 (viii) 1.018
2. (i) 0° (ii) 30° (iii) 120° (iv) 20°
3. (i) $m=\frac{2}{3}, \theta=33.69^\circ$ (ii) $m=-\frac{7}{3}, \theta=113.20^\circ$ (iii) $m=\frac{1}{7}, \theta=8.13^\circ$
4. (i) $m=-1$ (ii) $m=1$ 5. $x=0$ 6. $k=\frac{15}{8}$ 7. $k=15$ 9. $y=1$
12. $P(x, y)=P(7,-1)$ 13. Fourth vertex: $(-3,-2)$
14. Slopes of sides: $-1, \frac{3}{8}, \frac{8}{3}$; Slopes of medians: $-\frac{13}{2}, -\frac{2}{13}, 1$; Slopes of altitudes: $1, -\frac{8}{3}, -\frac{3}{8}$

Exercise 8.2

1. (i) $y=3$ (ii) $y=0$ (iii) $y=-9$ (iv) $y=-\frac{5}{2}$
2. (i) $x=1$ (ii) $x=9$ (iii) $x=-4$ (iv) $x=\frac{1}{4}$
3. (i) $2x-y-3=0$ (ii) $4x-y+27=0$ (iii) $y+5=0$
   (iv) $x+2=0$ (v) $5x+8y+22=0$ (vi) $4x-3y-20=0$
   (vii) $5x-6y+30=0$ (viii) $x+y-11=0$
4. (i) $\frac{x+4}{4/5}=\frac{y-2}{3/5}$ (ii) $\frac{x-6}{\cos 30^\circ}=\frac{y+6}{\sin 30^\circ}$ 5. (i) $x-\sqrt{3}y+10=0$ (ii) $x+y-2\sqrt{10}=0$
6. (i) $5x+y+24=0$ (ii) $4x+y-19=0$ (iii) $x-2y+8=0$
   (iv) $x+4y+2=0$ (v) $x+y-3=0$ (vi) $x-2y-14=0$
7. $4x-3y+9=0$ 8. $2x+y+5=0$ 9. $x-4y+7=0$
10. Equations of medians: $2x+y-4=0$, $x-y+2=0$, $7x-4y+6=0$
    Equations of altitudes: $3x+4y-16=0$, y $x+3$ $-22=0$, $2x+y+6=0$
11. (a) Slope-intercept form: $y=-\frac{3}{4}x+\frac{11}{8}$; Two-intercept form: $\frac{x}{11/6}+\frac{y}{11/8}=1$
    Point-slope form: $y-\frac{11}{8}=-\frac{3}{4}(x-0)$; Two-point form: $\frac{y-11/8}{0-11/8}=\frac{x-0}{11/6-0}$
    Normal form: $x\left(\frac{3}{5}\right)+y\left(\frac{4}{5}\right)=-\frac{3}{4}(x-0)$; Symmetric form: $\frac{x-0}{-4/5}=\frac{y-11/8}{3/5}$ (b) Do yourself.

Exercise 8.3

1. (i) 45° (ii) 104° (iii) 90° 2. (i) 59° (ii) 14° (iii) 18.4° (iv) 45° (v) 85.6°
3. (i) $45^\circ, 45^\circ, 90^\circ$ (ii) $31^\circ, 143^\circ, 6^\circ$ (iii) $37^\circ, 78^\circ, 65^\circ$ (iv) $45^\circ, 96^\circ, 39^\circ$
4. (i) 90° (ii) 53° (iii) 108°
5. (i) $90^\circ, 45^\circ, 45^\circ$ (ii) $54^\circ, 82^\circ, 44^\circ$ (iii) $63^\circ, 90^\circ, 27^\circ$ (iv) $59^\circ, 79^\circ, 42^\circ$
6. (i) $(1,-3)$ (ii) $\left(-\frac{23}{7}, \frac{2}{7}\right)$ (iii) $\left(\frac{13}{23}, -\frac{19}{23}\right)$ 7. (a) $x+1=0$ (b) $3x-4y+7=0$
8. $15x-5y+39=0$ 9. (a) $11y-7=0$ (b) $22x+9=0$
10. (a) $x-2y+1=0$ (b) $5x-2y+5=0$ 11. (a) $7x+2y-20=0$ (b) $2x-7y+17=0$

Exercise 8.4

1. $4x + 5y = 40$, 5 dozen **2.** (i) Rs.12150 (ii) 8 people

3. $y = 5500x + 700$, $b = 800a$; First deal is better.

4. (i) $y = 120x + 500$, total amount = Rs.1940 (ii) earning per hour (iii) additional amount

5. (i) $y = 18x + 44$ (ii) 584 units (iii) Rs.11680 (iv) 20 days

6. (i) $y = 900x + 1500$ (ii) Rs.6000 (iii) charge per hour

7. (i) $F = \frac{9}{5}C + 32$ (ii) 41°F (iii) y-intercept shows that temperature in F starts from 32 and slope shows rate of increase in temperature in F.

8. (i) $y = 6x$ (ii) run rate per over (iii) scores start from 0 (iv) 270 runs (v) 40 overs

9. (i) $(y - 5000) = 200(x - 0)$ (ii) Rs.200, slope shows the amount per kilometre (iii) Rs.41000

10. (i) $(y - 67) = -\frac{13}{7}(x - 25)$, 41° E

11. (i) $y = 2x$, Length of plot = 60 feet (ii) Slope is 2 which shows relation between length and width.

Miscellaneous Exercise 8

1. (i) c (ii) b (iii) d (iv) a (v) b (vi) a (vii) c (viii) d (ix) a (x) b (xi) c (xii) c (xiii) d (xiv) a

3. $y + \frac{x}{2} = 1$ **4.** $y = \frac{5}{2}x + \frac{1}{2}$, $\frac{x}{-1/5} + \frac{y}{1/2} = 1$

5. (i) $\frac{x}{3} + \frac{y}{-1} = 1$, $a = 3$, $b = -1$ (ii) $m = \frac{1}{3}$, $y + 1 = \frac{1}{3}(x - 0)$ **6.** $y = \sqrt{3}x + 20$, $m = \sqrt{3}$, $c = 20$

7. (i) A(– 4, 2), B(8, 5), C(1, 9) (ii) $m_1 = \frac{1}{4}$, $m_2 = -\frac{4}{7}$, $m_3 = \frac{7}{5}$ (iii) Interior angles: 40°, 44°, 96°

8. (i) M(6, 1) (ii) Slope of PM = 1 (iii) $x - y + 4 = 0$ (iv) $x + y = 0$

9. (i) $\sqrt{68}$ units (ii) Slope of BA = 4 (iii) $a = 4$, $b = -1$ (iv) $4x - y + 3 = 0$

10. $2x - 5y - 2 = 0$ **11.** (i) O(0, 0), C(5, 6), A(– 3, 5), B(5, 2)

(ii) Slope of OC $= \frac{6}{5}$, Slope of AB $= -\frac{3}{8}$ (iii) $6x - 5y = 0$, $3x + 8y - 31 = 0$ (iv) $\left(\frac{155}{63}, \frac{62}{21}\right)$

12. l: A(0, –1), B(2, 0) (i) Slope of $l = \frac{1}{2}$ (ii) Slope of $p = -2$

(iii) mid-point of $l = \left(1, -\frac{1}{2}\right)$ (iv) $4x + 2y - 3 = 0$

Exercise 9.1

2. (i), (ii), (iii), (v), (vi), (viii), (ix), (xi) **3.** No, by assigning particular values to a and b.

5. 63, 'the next number is 2 × 31 + 1 = 63'

6. Axioms: (i), (ii), (v), (vii), (viii), (x) Postulates: (iii), (vi), (ix)
Neither axioms nor postulate: (iv).

Exercise 9.2

1. Similar pairs: (i), (ii), (iv), (vi), (vii)
2. (i) $x = 10$ cm, $y = 6$ cm (ii) $a = 30°$ (iii) $x = 7.5$ cm (iv) $x = 2$ cm, $y = 3.6$ cm, $z = 8$ cm
   (v) $a = 4$ cm, $b = 4.2$ cm (vi) $a = 4$ cm, $b = 6$ cm (vii) $a = 12$ cm
3. DE = 3 cm, EF = 4 cm, FD = 2 cm 4. DE = 18 m 5. DE = 15 m 6. 20 cm 7. 10 m
8. (a) $a = 3$ cm (b) $x = 2.4$ cm, $y = 5.25$ cm (c) $a = 16$ cm (d) $x = 7.2$ cm, $y = 4.5$ cm
9. $\angle C = \angle AED = 50°$, yes 10. $\angle A = \angle B = 60°$, equilateral and acute angled 11. 10 m

Exercise 9.3

1. (i) 16 cm^2 (ii) 32 cm^2 (iii) 249 cm^2 (iv) 45 cm^2
2. (i) 6 cm (ii) 5 cm (iii) 30 cm (iv) 19.6 cm
3. (i) 9 : 16 (ii) 640 cm^2 (iii) 900 cm^2 4. 4 : 5, 10 cm 5. XB : XZ = AB : YZ = 5 : 6
6. 5600 m^2 7. (i) 9 : 1 (ii) 3150 cm^2
8. (i) 8 : 13 each (ii) 64 : 169 (iii) 192 cm^2 (iv) 315 cm^2, trapezium

Exercise 9.4

1. (i) yes (ii) no (iii) yes (iv) no
2. (i) d = 20 cm, $V_1 : V_2 = 125 : 1$ (ii) h = 9 m, c = 9.5 m, b = 18 m, $V_1 : V_2 = 27 : 8$
3. (i) $V_2 = 333.33$ mm^3 (ii) $V_1 = 208.33$ ft^3 (iii) $V_1 = 29.44$ cm^3 (ii) $V_2 = 121.5$ m^3
4. 8 : 27 5. 5.8 cups approx 6. 10000 ft^3 7. Area = 292.5π cm^2, Volume = 945π cm^3
8. 2 : 5 9. 15 cm 10. 625 cm^3 11. (i) 4 : 5 (ii) 16 : 25 (iii) 64 : 125
12. 6 cm 13. 3 m

Exercise 9.5

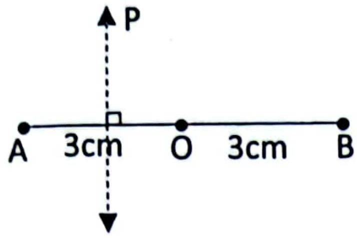
1. (a) 8 (b) 6 (c) 3 (d) 9 2. (a) k = 108° (b) p = q = r = s = t = 72°
3. 12 sides 4. Minimum interior angle = 60°, maximum interior angle = 179.99999….(< 180°)
   If maximum interior angle is 180°, then adjacent sides of polygon become collinear.
5. (a) 72° (b) 40° (c) 24° (d) 18° 6. (a) 60° (b) 120° (c) 6
7. No, because 360° is not a multiple of 50°. 8. square 9. decagon 10. 5040° 12. 5 sides
13. 28°, 49°, 71°, 93°, 119° 14. 7 15. 1980° 16. 18 sides

Exercise 9.6

1. P = 15 m, cost = Rs.3,300 2. Rs.27,300 3. P = 30 m, A = 60 m^2
4. (i) Rs.9,600 (ii) 259.81 m^2 (iii) Rs.129,900
5. (i) Rs.50,400 (ii) 120 ft^2 (iii) Rs.4,800 6. P = 14 ft

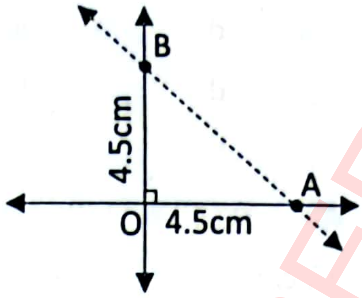
Exercise 9.7

1.



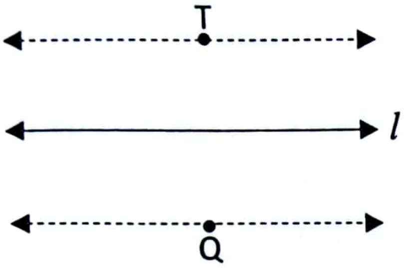
Locus is perpendicular bisector of $\overline{AO}$.

2.



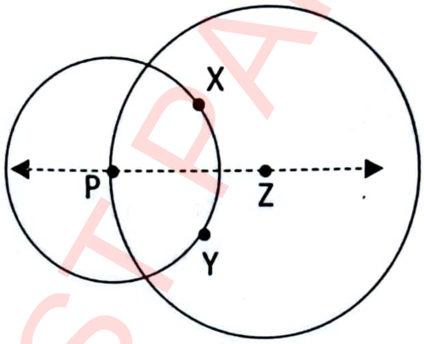
Locus: Line making equal intercepts

3.



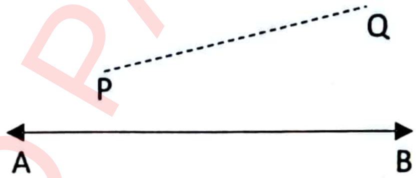
(iii) 6.2 cm

4.

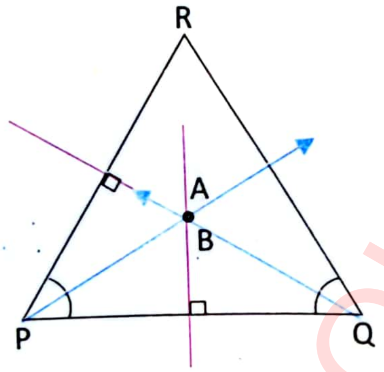


(iii) Second circle passes through the centre

5. (i) line segment (ii) yes
(iii) by drawing $\overline{PQ}$ parallel to $\overleftrightarrow{AB}$.
(iv) by taking P and Q on $\overleftrightarrow{AB}$.

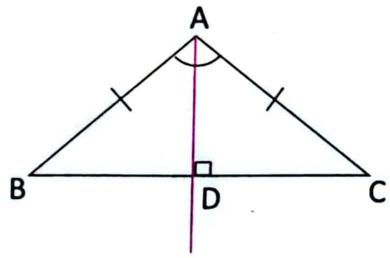


6.



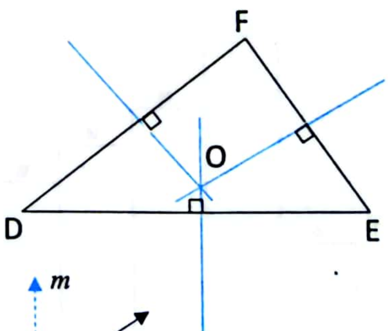
(iii) Points A and B coincide.

7.



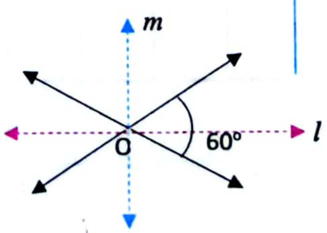
By measuring angle ADC, and sides BD and CD, we see that AD is right bisector of BC.

8. Join non collinear points D, E and F.
Find point of intersection of right bisector of sides.
Locus of point equidistant from given points is O.
Only one such point exists.



9. (iii) Locus l and angle bisector are same.

(iv) Both angle bisectors are perpendicular.



Miscellaneous Exercise 9

1. (i) c (ii) d (iii) a (iv) b (v) c (vi) c
(vii) d (viii) d (ix) b (x) d (xi) a (xii) b
(xiii) c (xiv) b (xv) d (xvi) a (xvii) a

2. 60° 3. No 4. yes 5. 140 6. (ii) 5 cm 7. 32 cm

9. BC = 8 cm, DE = 12 cm 10. 1 : 9 11. 200 cm^2 12. 135 ft^3

13. (i) 756 m^2 (ii) 1012.5 in^2

Exercise 11.1

1. (i) 30 (ii) 31 – 40 (iii) none (iv) 51 (v) 50.5 (vi) 10 (vii) 5.5, 15.5, 25.5, 35.5, 45.5, 55.5

2. (i), (ii)

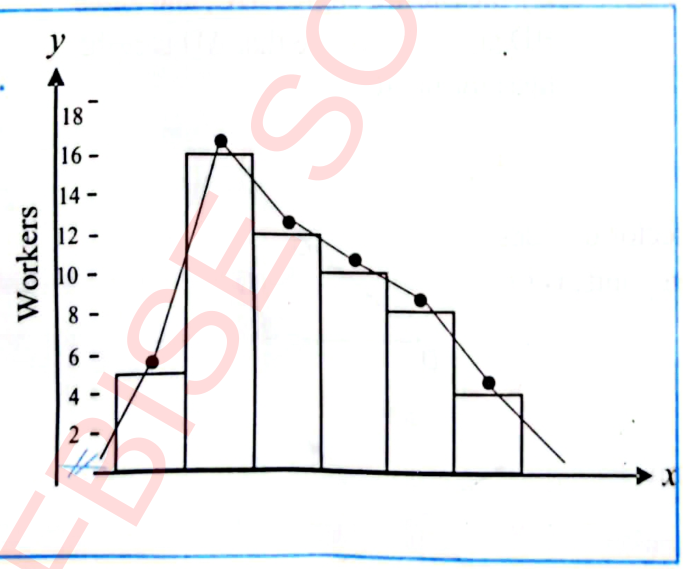
Class Interval (Cards made)	Frequency (No. of students)	Cumulative frequency
40 – 49	7	7
50 – 59	13	20
60 – 69	22	42
70 – 79	17	59
80 – 89	13	72
90 – 99	8	80

(iii). 42 (iv). 8.75% (v). 60-69 (vi). 10

3.

Medals won	0	1	2	3	4	5	6
No. of players Frequency(f)	5	8	10	8	5	5	4
Cumulative frequency	5	13	23	31	36	41	45

4.



5. Do yourself

6. Do Yourself 7. (i) 48 (ii) 8 (iii) 1 (iv) 48% app (v) 8

(vi)

C.I	9-11	11-13	13-15	15-17	17-19	19-21	21-23
f	8	9	13	10	5	2	1

8. Do Yourself 9. Do Yourself

8

Exercise 11.2

1. (i) 7 (ii) 0 (iii) 14 (iv) 6.95 (v) 5 2. $\bar{x}_A = 48,\ \bar{x}_B = 33.6$, A is better

3. 78, 67.5, 69, 77 4. 11 .33 5. 23.51 6. (i) 4 (ii) 0 (iii) 2 (iv) 2 (v) 4

7. 1.0 8. 64.09 9. (i) no (ii) 2 (iii) 120, 225 (iv) no

10. mean = median = mode = 2 11. (a) 9.83 (b) 14.38

Exercise 11.3

1. (i) 2/5 (ii) 1/5 (iii) 2/5 (iv) 3/5 2. (i) 1/4 (ii) 5/12 (iii) 0.00175

3. (i) 1/8 (ii) 1/4 (iii) 7/8 (iv) 0 (v) 1

4. (i) 2/5 (ii) 7/10 (iii) 3/5 (iv) 9/10 (v) 3/10 (vi) 3/10 (vii) 1/5 (viii) 2/5 (ix) 1/2

5. (i) 1/2 (ii) 0 (iii) 9/10 (iv) 1/10 (v) 1/10 (vi) 1/10 (vii) 3/5 (viii) 3/10

6. relative frequencies 3/5, 2/5 (i) 312 (ii) 122 7. relative frequencies 1/8, 5/24, 1/8, 1/6, 1/4, 1/8 (i) 750 (ii) 375 (iii) 83 app 8. (a) 1/6 (b) 5/6 (d) 1 (e) 10

Miscellaneous Exercise 11

1. (i). (b) (ii). (a) (iii). (c) (iv). (d) (v). (c) (vi). (a) (vii). (b) (viii). (c) (ix). (b) (x). (b) (xi). (c) (xii). (b) (xiii). (c) (xiv). (b) (xv). (d) (xvi). (b) (xvii). (d) (xviii). (d) (xix). (c) (xx). (a)

2. a. mean 23.44, median = mode 25 3. 53:59 4. 35 5. a. mean = median = mode = 25 6. (a) 50 hits (b) 450 misses 7. 8 days 8 16 assessments 9. (a) 13/50 (b) 37/50 (c) 11/50

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170						4	9	13	17	21	26	30	34	38
						0212	0253	0294	0334	0374	4	8	12	16	20	24	28	32	36
11	0414	0453	0492	0531	0569						4	8	12	15	19	23	27	31	35
						0607	0645	0682	0719	0755	4	7	11	15	19	22	26	30	33
12	0792	0828	0864	0899	0934						3	7	11	14	18	21	25	28	32
						0969	1004	1038	1072	1106	3	7	10	14	17	20	24	27	31
13	1139	1173	1206	1239	1271						3	7	10	13	16	20	23	26	30
						1303	1335	1367	1399	1430	3	7	10	13	16	19	22	25	29
14	1461	1492	1523	1553	1584						3	6	9	12	15	19	22	25	28
						1614	1644	1673	1703	1732	3	6	9	12	15	17	20	23	26
15	**1761**	**1790**	**1818**	**1847**	**1875**						**3**	**6**	**9**	**11**	**14**	**16**	**20**	**23**	**26**
						1903	**1931**	**1959**	**1987**	**2014**	**3**	**6**	**8**	**11**	**14**	**17**	**19**	**22**	**24**
16	2041	2068	2095	2122	2148						3	5	8	11	14	17	19	22	24
						2175	2201	2227	2253	2279	3	5	8	10	13	16	18	21	23
17	2304	2330	2355	2380	2405						3	5	8	10	13	15	18	20	23
						2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648						2	5	7	9	12	14	16	19	21
						2672	2695	2718	2742	2765	2	5	7	9	11	14	16	18	21
19	2788	2810	2833	2856	2878						2	4	7	9	11	13	16	18	20
						2900	2923	2945	2967	2989	2	4	6	8	11	13	15	17	19
20	**3010**	**3032**	**3054**	**3075**	**3096**	**3118**	**3139**	**3160**	**3181**	**3201**	**2**	**4**	**6**	**8**	**11**	**13**	**15**	**17**	**19**
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	**3979**	**3997**	**4014**	**4031**	**4048**	**4065**	**4082**	**4099**	**4116**	**4133**	**2**	**3**	**5**	**7**	**9**	**10**	**12**	**14**	**15**
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	**4771**	**4786**	**4800**	**4814**	**4829**	**4843**	**4857**	**4871**	**4886**	**4900**	**1**	**3**	**4**	**6**	**7**	**9**	**10**	**11**	**13**
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	**5441**	**5453**	**5465**	**5478**	**5490**	**5502**	**5514**	**5527**	**5539**	**5551**	**1**	**2**	**4**	**5**	**6**	**7**	**9**	**10**	**11**
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	**6021**	**6031**	**6042**	**6053**	**6064**	**6075**	**6085**	**6096**	**6107**	**6117**	**1**	**2**	**3**	**4**	**5**	**6**	**8**	**9**	**10**
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	4635	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	**6532**	**6542**	**6551**	**6561**	**6571**	**6580**	**6590**	**6599**	**6609**	**6618**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
50	**6990**	**6998**	**7007**	**7016**	**7024**	**7033**	**7042**	**7050**	**7059**	**7067**	**1**	**2**	**3**	**3**	**4**	**5**	**6**	**7**	**8**
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
55	**7404**	**7412**	**7419**	**7427**	**7435**	**7443**	**7451**	**7459**	**7466**	**7474**	**1**	**2**	**2**	**3**	**4**	**5**	**5**	**6**	**7**
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	**7782**	**7789**	**7796**	**7803**	**7810**	**7818**	**7825**	**7832**	**7839**	**7846**	**1**	**1**	**2**	**3**	**4**	**4**	**5**	**6**	**6**
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	**8129**	**8136**	**8142**	**8149**	**8156**	**8162**	**8169**	**8176**	**8182**	**8189**	**1**	**1**	**2**	**3**	**3**	**4**	**5**	**5**	**6**
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	**8451**	**8457**	**8463**	**8470**	**8476**	**8482**	**8488**	**8494**	**8500**	**8506**	**1**	**1**	**2**	**2**	**3**	**4**	**4**	**5**	**6**
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8696	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8730	8745	1	1	2	2	3	4	4	5	5
75	**8751**	**8756**	**8762**	**8768**	**8774**	**8779**	**8785**	**8791**	**8797**	**8802**	**1**	**1**	**2**	**2**	**3**	**3**	**4**	**5**	**5**
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	**9031**	**9036**	**9042**	**9047**	**9053**	**9058**	**9063**	**9069**	**9074**	**9079**	**1**	**1**	**2**	**2**	**3**	**3**	**4**	**4**	**5**
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	**9294**	**9299**	**9304**	**9309**	**9315**	**9320**	**9325**	**9330**	**9335**	**9340**	**1**	**1**	**2**	**2**	**3**	**3**	**4**	**4**	**5**
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	**9542**	**9547**	**9552**	**9557**	**9562**	**9566**	**9571**	**9576**	**9581**	**9586**	**0**	**1**	**1**	**2**	**2**	**3**	**3**	**4**	**4**
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9603	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	**9777**	**9782**	**9786**	**9791**	**9795**	**9800**	**9805**	**9809**	**9814**	**9818**	**0**	**1**	**1**	**2**	**2**	**3**	**3**	**4**	**4**
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4

TABLES OF ANTILOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
.00	**1000**	**1002**	**1005**	**1007**	**1009**	**1012**	**1014**	**1016**	**1019**	**1021**	**0**	**0**	**1**	**1**	**1**	**1**	**2**	**2**	**2**
.01	1023	1026	1027	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
.02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
.03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
.04	1096	1099	1102	1104	1107	1100	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
.05	**1122**	**1125**	**1127**	**1130**	**1132**	**1135**	**1138**	**1140**	**1143**	**1146**	**0**	**1**	**1**	**1**	**1**	**2**	**2**	**2**	**2**
.06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
.07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
.08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
.09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
.10	**1259**	**1262**	**1265**	**1268**	**1271**	**1274**	**1276**	**1279**	**1282**	**1285**	**0**	**1**	**1**	**1**	**1**	**2**	**2**	**2**	**3**
.11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
.12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
.13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
.14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
.15	**1413**	**1416**	**1419**	**1422**	**1426**	**1429**	**1432**	**1435**	**1439**	**1442**	**0**	**1**	**1**	**1**	**2**	**2**	**2**	**3**	**3**
.16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
.17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
.18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
.19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
.20	**1585**	**1589**	**1592**	**1596**	**1600**	**1603**	**1607**	**1611**	**1614**	**1618**	**0**	**1**	**1**	**1**	**2**	**2**	**3**	**3**	**3**
.21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
.22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
.23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
.24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
.25	**1778**	**1782**	**1786**	**1791**	**1795**	**1799**	**1803**	**1807**	**1811**	**1816**	**0**	**1**	**1**	**2**	**2**	**2**	**3**	**3**	**4**
.26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
.27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
.28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
.29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
.30	**1995**	**2000**	**2004**	**2009**	**2014**	**2018**	**2023**	**2028**	**2032**	**2037**	**0**	**1**	**1**	**2**	**2**	**3**	**3**	**4**	**4**
.31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
.32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
.33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
.34	2188	2193	2198	2103	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
.35	**2239**	**2244**	**2249**	**2254**	**2259**	**2265**	**2270**	**2275**	**2280**	**2286**	**1**	**1**	**2**	**2**	**3**	**3**	**4**	**4**	**5**
.36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
.37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
.38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
.39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
.40	**2512**	**2518**	**2523**	**2529**	**2535**	**2541**	**2547**	**2553**	**2559**	**2564**	**1**	**1**	**2**	**2**	**3**	**4**	**4**	**5**	**5**
.41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
.42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2686	1	1	2	2	3	4	4	5	6
.43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
.44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
.45	**2818**	**2825**	**2831**	**2838**	**2844**	**2851**	**2858**	**2864**	**2871**	**2877**	**1**	**1**	**2**	**3**	**3**	**4**	**5**	**5**	**6**
.46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
.47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
.48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
.49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
.50	**3162**	**3170**	**3177**	**3184**	**3192**	**3199**	**3206**	**3214**	**3221**	**3228**	**1**	**1**	**2**	**3**	**4**	**4**	**5**	**6**	**7**
.51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
.52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
.53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
.54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
.55	**3548**	**3556**	**3565**	**3573**	**3581**	**3589**	**3597**	**3606**	**3614**	**3622**	**1**	**2**	**2**	**3**	**4**	**5**	**6**	**7**	**7**
.56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
.57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
.58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
.59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
.60	**3981**	**3990**	**3999**	**4009**	**4018**	**4027**	**4036**	**4046**	**4055**	**4064**	**1**	**2**	**3**	**4**	**5**	**6**	**6**	**7**	**8**
.61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
.62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
.63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
.64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
.65	**4467**	**4477**	**4487**	**4498**	**4508**	**4519**	**4529**	**4539**	**4550**	**4560**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
.66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
.67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
.68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
.69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
.70	**5012**	**5023**	**5035**	**5047**	**5058**	**5070**	**5082**	**5093**	**5105**	**5117**	**1**	**2**	**4**	**5**	**6**	**7**	**8**	**9**	**11**
.71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
.72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
.73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
.74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
.75	**5623**	**5636**	**5649**	**5662**	**5675**	**5689**	**5702**	**5715**	**5728**	**5741**	**1**	**3**	**4**	**5**	**7**	**8**	**9**	**10**	**12**
.76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
.77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
.78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
.79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
.80	**6310**	**6324**	**6339**	**6353**	**6368**	**6383**	**6397**	**9412**	**6427**	**6442**	**1**	**3**	**4**	**6**	**7**	**9**	**10**	**12**	**13**
.81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
.82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
.83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
.84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
.85	**7079**	**7096**	**7112**	**7129**	**7145**	**7161**	**7178**	**7194**	**7211**	**7228**	**2**	**3**	**5**	**7**	**8**	**10**	**12**	**13**	**15**
.86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
.87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
.88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
.89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
.90	**7943**	**7962**	**7980**	**7998**	**8017**	**8035**	**8054**	**8072**	**8091**	**8110**	**2**	**4**	**6**	**7**	**9**	**11**	**13**	**15**	**17**
.91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
.92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
.93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
.94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
.95	**8913**	**8933**	**8954**	**8974**	**8995**	**9016**	**9036**	**9057**	**9078**	**9099**	**2**	**4**	**6**	**8**	**10**	**12**	**15**	**17**	**19**
.96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
.97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
.98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
.99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

GLOSSARY

Absolute value: The absolute value of a number is its distance from 0 on the number line.
Absolute value equation: An equation that contains an absolute value symbol.
Additive identity: Additive identity of real numbers is zero but additive identity of matrices are null matrices.
Additive inverse: The element which is inverse to a given element with respect to addition.
Algebraic expression: An expression consisting of variables and constants, connected by algebraic operations.
Altitude: A line segment between a vertex and a side of a polygon, that is perpendicular to the side.
Angle bisector: A ray that divides an angle into two equal parts.
Anti logarithm: Inverse operation of logarithm.
Arithmetic mean: It is ratio of sum of quantities and number of quantities.
Axiom: It is a mathematical statement which is assumed to be true without any proof.
Base of logarithm: The number in terms of which a given number is expressed as a logarithm or exponential.
Bearings: Bearings are the angles measured clockwise from due north and are expressed as three digit numbers.
Binary relation: A binary relation R is a subset of the Cartesian product of two sets.
Cartesian plane: the two dimensional, the points of which are identify by their coordinates.
Cartesian product of sets: Cartesian product of sets A and B is the set of ordered pairs (x, y) such that $x \in A$ and $y \in B$.
Characteristic: The integral part of logarithm of a number.
Circular region: The union of a circle and its interior region is the circular region.
Circular system: The system in which angle is measured in radians.
Coefficient of variation: The percentage ratio between standard deviation and mean of a data.
Collinear points: The points lying on same straight line.
Common logarithm: Logarithm having base 10.
Complement of set: Let $A \subset U$, then the set of all elements of U, which are not in A, is called complement of A.
Complementary events: Two events in which one shows non-occurrence of other.
Compound inequalities: Two inequalities that are joined by the word "and" or the word "or".
Conjecture: A statement that is believed to be true but its truth has not been proved. It is a true statement that needs proof.
Converse: A proposition derived from another by interchanging its subject and predicate terms.
Corollary: A proposition that follows directly from the statement of another proposition.
Coterminal angles: Two angles having same terminal ray in standard position.
Deductive reasoning: In deductive reasoning we work from some general result to a specific conclusion.
Diagonal: A line joining any two vertices of a polygon that are not joined by any of its edges.
Difference of sets: Difference of sets A and B is a collection of those elements of A which are not present in B.
Disjoint sets: Two sets are said to be disjoint if they have no common elements.
Distance formula: It provides distance between two points in a plane or line.
Domain of binary relation: Set of the first elements of ordered pairs in a binary relation.
Equality: A mathematical statement with the equality sign is called an equality.
Equivalent equations: Two or more equations, having the same solution.
Equivalent inequalities: Two or more inequalities that have the same solution.
Expected frequency: The expected frequency of an event is the frequency we expect to see based on probabilities.
Exterior angle of polygon: Angle formed by any side of the polygon and the extension of its adjacent side.
Extraneous root: A solution of equation which does not satisfy solution of equation.

Factor: Any integer or polynomial that exactly divides a given integer or polynomial.

Factor theorem: A polynomial p(x) has a factor $x - c$, if and only if p(c) = 0.

Factorization of algebraic expression: The process in which an algebraic expression can be expressed as the product of its factors.

Family of lines: For a non-zero k, the equation $l_1 + kl_2 = 0$, is also linear and represents family of lines where l_1 and l_2 are lines.

Frequency: Number of repetitions of an individual value in an ungrouped data.

Frequency table: A table consisting of values of a data along with their frequencies is called a frequency table.

Fundamental assumptions: Fundamental assumptions are statements which are regarded true without any proof.

Geometric mean: It is a measure of central tendency which is usually used to find the average rates of change in different intervals.

Gradient: It is the tangent of angle which the line makes with positive direction of the x-axis.

Grouped data: A tabular arrangement of data in which various items are arranged or distributed into some classes.

Harmonic mean: It is ratio of number of values to the sum of reciprocal of values of data.

Histogram: The representation of data by a chart having adjacent rectangular bars.

Identity: It is an equation which remains true for all possible replacements of variable involved in it.

Inclination of straight line: Inclination of a straight line is the angle

SYMBOLS AND ABBREVIATIONS USED IN MATH

$=$	→	is equal to
$\neq$	→	is not equal to
$\in$	→	is member of
$\notin$	→	is not member of
$\emptyset$	→	empty set
$\cup$	→	union of sets
$\cap$	→	intersection of sets
$\Leftrightarrow$	→	if and only if
$\overline{AB}$	→	line Segment AB
AB	→	measurement of side AB
$\angle A$	→	measurement of angle A
$\cong$	→	is congruent to
$\perp$	→	is perpendicular to
Δ	→	triangle
$\Rightarrow$	→	implies that
$\wedge$, &	→	and
$\vee$	→	or
$<$	→	is less than
$>$	→	is greater than
$\leq$	→	is less than or equal to
$\geq$	→	is greater than or equal to
@	→	at the rate of
%	→	percent
π	→	Pie
:	→	ratio
::	→	proportion
$\therefore$	→	therefore, hence
$\because$	→	because, since
i.e.	→	that is
$\approx$	→	approximately equal to
$\sqrt{\ }$	→	square root / radical
e.g.	→	for example
/	→	such that
$\leftrightarrow$	→	corresponding to
//	→	is parallel to

I N D E X

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قومی ترانہ

پاک سرزمین شاد باد! کشورِ حسین شاد باد!
تو نشانِ عزمِ عالی شان ارضِ پاکستان
مرکزِ یقین شاد باد!

پاک سرزمین کا نظام قوّتِ اخوّتِ عوام
قوم، ملک، سلطنت پائندہ تابندہ باد!
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